মানুষের জ্ঞান ও ভাবকে বইয়ের মধ্যে সঞ্চিত করিবার যে একটা প্রচুর সুবিধা আছে, সে কথা কেহই অস্বীকার করিতে পারে না। কিন্তু সেই সুবিধার দ্বারা মনের স্বাভাবিক শক্তিকে একেবারে আচ্ছন্ন করিয়া ফেলিলে বুদ্ধিকে বাবু করিয়া তোলা হয়।

— রবীন্দ্রনাথ ঠাকুর

ভারতের একটা mission আছে, একটা গৌরবময় ভবিষ্যৎ আছে, সেই ভবিষ্যৎ ভারতের উত্তরাধিকারী আমরাই। নূতন ভারতের মুক্তির ইতিহাস আমরাই রচনা করছি এবং করব। এই বিশ্বাস আছে বলেই আমরা সব দুঃখ কষ্ট সহ্য করতে পারি, অন্ধকারময় বর্তমানকে অগ্রাহ্য করতে পারি, বাস্তবের নিষ্ঠুর সত্যগুলি আদর্শের কঠিন আঘাতে ধূলিসাৎ করতে পারি।

— সুভাষচন্দ্র বসু

Any system of education which ignores Indian conditions, requirements, history and sociology is too unscientific to commend itself to any rational support.

— Subhas Chandra Bose

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NETAJI SUBHAS OPEN UNIVERSITY

Choice Based Credit System (CBCS)

SELF LEARNING MATERIAL

HMT

MATHEMATICS

Theory of Real Functions and Functions of Several Variables

CC-MT-08

Under Graduate Degree Programme

PREFACE

In a bid to standardise higher education in the country, the University Grants Commission (UGC) has introduced Choice Based Credit System (CBCS) based on five types of courses viz. *core, discipline specific, generic elective, ability* and *skill enhancement* for graduate students of all programmes at Honours level. This brings in the semester pattern, which finds efficacy in sync with credit system, credit transfer, comprehensive continuous assessments and a graded pattern of evaluation. The objective is to offer learners ample flexibility to choose from a wide gamut of courses, as also to provide them lateral mobility between various educational institutions in the country where they can carry acquired credits. I am happy to note that the University has been accredited by NAAC with grade 'A'.

UGC (Open and Distance Learning Programmes and Online Learning Programmes) Regulations, 2020 have mandated compliance with CBCS for U.G. programmes for all the HEIs in this mode. Welcoming this paradigm shift in higher education, Netaji Subhas Open University (NSOU) has resolved to adopt CBCS from the academic session 2021-22 at the Under Graduate Degree Programme level. The present syllabus, framed in the spirit of syllabi recommended by UGC, lays due stress on all aspects envisaged in the curricular framework of the apex body on higher education. It will be imparted to learners over the *six* semesters of the Programme.

Self Learning Materials (SLMs) are the mainstay of Student Support Services (SSS) of an Open University. From a logistic point of view, NSOU has embarked upon CBCS presently with SLMs in English / Bengali. Eventually, the English version SLMs will be translated into Bengali too, for the benefit of learners. As always, all of our teaching faculties contributed in this process. In addition to this we have also requisitioned the services of best academics in each domain in preparation of the new SLMs. I am sure they will be of commendable academic support. We look forward to proactive feedback from all stakeholders who will participate in the teaching-learning based on these study materials. It has been a very challenging task well executed, and I congratulate all concerned in the preparation of these SLMs.

I wish the venture a grand success.

Professor (Dr.) Subha Sankar Sarkar
Vice-Chancellor

Netaji Subhas Open University

Under Graduate Degree Programme
Choice Based Credit System (CBCS)
Subject : Honours in Mathematics (HMT)
Course : Theory of Real Functions and Functions of Several Variables
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Netaji Subhas Open University

UG Mathematics (HMT)

Subject : Honours in Mathematics (HMT)

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Unit-1 □ Limits of Functions

Structure

1.0 Objectives

This unit gives

- Various types of functions and their classification
- Sequence of real number and its convergence
- Concept of limit of a real function
- Various properties of limit of a function such as algebric operation on limits, sandwich property, etc.

1.1 Introduction

The limit of a function is a fundamental concept in analysis concerning the behaviour of that function near a particular point. Although implicit in the development of calculus of the 17th & 18th centuries, the modern idea of the limit of a function goes back to Bolzano who, in 1817, introduced the basic of the epsilon-delta technique to define limit of functions. The motion of a limit has many applications in modern Calculus. In particular, the many definitions of continuity employ the limit. It also appears in the definition of the derivative.

1.2 Pre requisites

(or Recapitulation of prior elementary ideas that are needed to introduce the concept of limit):

A. Functions

(i) Let A and B be two non-void subsets $\mathbb{R}$ & $f : A \to B$ is a rule of correspondence that assigns to each $x \in A$, a uniquely determined $y \in B$ or $y = f(x)$.

The set of values of x for which f can be defined is known as **Domain** of f, denoted by D_f and the corresponding collection of y's (as mentioned above) is known as Range set of f generally denoted by R_f.

A few examples of f, D_f and R_f :

(i) $$f(x) = \sqrt{\left[\log_c \frac{5x - x^2}{4}\right]}$$

f can be defined for those x for which $\frac{5x - x^2}{4} \geq 1$ and this gives $1 \leq x \leq 4$

so $D_f \equiv [1, 4]$

(ii) $$f(x) = \sqrt{\left(x - \frac{x}{1-x}\right)}$$

f can be defined only when $x - \frac{x}{1-x} \geq 0 \Rightarrow 1 < x < \infty$ & $D_f = (1, \infty)$

(iii) $f(x) = \cos^{-1} \frac{3}{4 + 2\sin x}$. Here we must have $-1 \leq \frac{3}{4 + 2\sin x} \leq 1$

& for this $D_f \equiv \left[-\frac{\pi}{6} + 2k\pi, \frac{7\pi}{6} + 2k\pi\right]$ where $k = 0, \pm 1, \pm 2$,......

Note that D_f may be a closed and bounded interval, may be an open interval (bounded or unbounded), union of intervals and so on.

(Readers are requested to verify the validity of D_f as mentioned in above examples and as well as to look for other functions and their domain).

(i) Consider the function $f:[-1,\ 1]\to\mathbb{R}$ defined by $f(x)=\begin{cases}\dfrac{|x|}{x}, & x\neq 0\\ 0, & x=0\end{cases}$

Here D_f is an interval $[-1,\ 1]$ but $R_f=\{-1,0,1\}$ which is not an interval.

(ii) Consider the function $f:(-1,\ 1)\to\mathbb{R}$ defined by $f(x)=\dfrac{1}{x^2+1}\ \forall\ x\in(-1,1)$

$$R_f=\left(\frac{1}{2},\ 1\right] \text{ or } \frac{1}{2}<x\leq 1.$$

Note that in D_f, -1 and $+1$ are not included but 1 is included as right hand end point in R_f.

We are interested to learn the reason for such differences of nature of D_f & R_f.

Equal functions : $f,\ g : D\to\mathbb{R}$ are same (or equal) when $f(x)=g(x)$ for each $x\in D$.

Note that x and $\dfrac{x^2}{x}$ are not same.

Operations on Functions : Let f and g be two functions having domain $D_f(\subset\mathbb{R})$ and $D_g(\subset\mathbb{R})$ respectively. If $D_f\cap D_g\neq\phi$, then $f\pm g, fg$ can be defined on $D_f\cap D_g$ by

(i) $(f\pm g)x=f(x)\pm g(x)\ \ \forall x\in D_f\cap D_g$ and

$(fg)(x) = f(x)\ g(x)\ \ \forall x \in D_f \cap D_g$

Again deleting those points of D_g (if any) for which $g(x) = 0$, we can define $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ where $x \in D_f \cap D_g \setminus \{x : g(x) = 0\}$.

Composition of functions : Let f and g be two functions such that

$x \in D_f \Rightarrow f(x) \in D_g$. In other words $R_f \subset D_g$. Then we can define

$(g \circ f)(x) = g[f(x)] \forall x \in D_f$.

$g \circ f$ is called the composite of two functions f and g.

Similarly, we can define $(f \circ g)(x)$ with appropriate restrictions.

In general $(f \circ g)(x) \neq (g \circ f)(x)$. For example, $f(x) = x^2, g(x) = \sin x$

Then $(g \circ f)(x) = g(f(x)) = \sin x^2$ & $(f \circ g)(x) = f(g(x)) = f(\sin x) = \sin^2 x$.

Injective (one-one), Surjective (onto) and Bijective functions :

Let $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}$.

If for $x, y \in D$, $f(x) = f(y) \Rightarrow x = y$, f is called injective or one-one function

$f(x) = 3x + 4,\ x \in \mathbb{R}$ is Injective but $g(x) = |x|,\ x \in \mathbb{R}$ is not Injective.

Let $f : D \to E$ where $D, E \subset \mathbb{R}$, obviously $f(D) \subseteq E$. But if $f(D) = E$, we say that f is surjective or onto function. $f : [1, 2] \to [2, 3]$ defined by $f(x) = x + 1$ is onto function.

But $f : [1, 2] \to [2,\ 4]$, $f(x) = x + 1$ is not so,

$$\frac{7}{2} \in [2,\ 4] \text{ and } \frac{7}{2} = x + 1 \Rightarrow x = \frac{5}{2} \notin [1,\ 2].$$

f is bijective if it is both injective and surjective.

Invertible functions : Let $f : X \to Y$ where $X, Y \subset \mathbb{R}$ be such that for each $y \in Y$, there exists a single value of x such that $f(x) = y$. Then this correspondence defines a function $x = g(y)$. We say that f is invertible and $x = g(y)$ is the inverse function. Note that if f be bijective, then f is invertible.

For example, if $y = \log_a\left(x + \sqrt{x^2+1}\right)$, $a > 0, a \neq 1$, then

$$x = \frac{1}{2}\left(a^y - a^{-y}\right) \text{ or } \sinh(y \ln a)$$

Increasing function & Decreasing function :

Let $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}$. If for each pair $x, y \in D$,

$x > y \Rightarrow f(x) \geq f(y)$ or $f(x) > f(y)$, we say that f is increasing function.

But if $x > y \Rightarrow f(x) \leq f(y)$ or $f(x) < f(y)$, we say that f is decreasing function.

$f(x) = \sin x$ is increasing in $\left[0, \frac{\pi}{2}\right]$ but is decreasing in $\left[\frac{\pi}{2}, \pi\right]$

Periodic function :

A function $f : D \to \mathbb{R}(D \subset \mathbb{R})$ is periodic if there exists a number p such that $f(x+p) = f(x) \forall x \in D$.

The smallest positive p for which $f(x+p) = f(x) \forall x$ holds, is called the period of f.

Bounded and unbounded functions :

$f : D \to \mathbb{R}(D \subset \mathbb{R})$ is said to be bounded above if there exists $\lambda \in \mathbb{R}$ such that $f(x) \leq \lambda \forall x \in D$, we say that f is bounded above (by λ). If there exists $\mu \in \mathbb{R}$ such that $f(x) \geq \mu \forall x \in D$, we say that f is bounded below (by μ). If f be both bounded above & bounded below, then *f* is bounded on $D_f\,(\equiv D)$. In other words, If there exists $K \in \mathbb{R}$ such that $|f(x)| \le K$ for all $x \in D$, we say that *f* is bounded on *D*. For future course of discussion the following concepts are useful.

Let $f : D \to \mathbb{R}\,(D \subset \mathbb{R})$ be bounded above.

Then $\lambda(\in \mathbb{R})$ is said to be the least upper bound or supremum of *f* in *D* if $\exists\ \lambda \in \mathbb{R}$ such that (i) $f(x) \le \lambda \forall x \in D$ and (ii) for any $\varepsilon > 0, \exists\, y \in D$ such that $f(y) > \lambda - \varepsilon$ (or in other words, no real $< \lambda$ is an upper bound of *f*) this $\lambda =$ sup f. If *f* be bounded above, then sup *f* $(\in \mathbb{R})$ exists.

If *f* is unbounded above we say that sup $f = \infty$

Let $f : D \to \mathbb{R}$ be bounded below. Then $\mu(\in \mathbb{R})$ is greatest lower bound or infimum of *f* in *D* if

(i) $f(x) \ge \mu$ for all $x \in D$ & (ii) if for any $\varepsilon > 0, \exists\, y \in D$ such that $f(y) < \mu + \varepsilon$, then μ = inf *f* (in other words, no real $> \mu$ is lower bound of *f*). Then μ = inf *f*. If *f* be bounded below, then inf *f* $(\in \mathbb{R})$ exists.

If *f* be unbounded below, we write inf $f = -\infty$

Sup *f* – inf *f* is known as oscillation of function *f* on *D*.

1.3 Sequences in $\mathbb{R}$

(i) *A* function $f : \mathbb{N} \to \mathbb{R}$ is known as a sequence (note that $\mathbb{N}$ is the set of natural numbers).

Examples : $\left\{(-1)^n\right\}_n, \left\{\dfrac{1}{n}\right\}_n, \left\{\dfrac{4n+3}{3n+4}\right\}, \left\{n^2\right\}_n$ etc.

Symbolically, $\{a_n\}_n\ (n \to a_n)$. Note that the range set of $\left\{(-1)^n\right\}_n$ is the set $\{-1, 1\}$ where as the range sets of the next three are infinite sets.

A sequence $\{a_n\}_n$ is bounded if its range set is bounded.

Range sets of $\{(-1)^n\}_n, \left\{\frac{1}{n}\right\}_n, \left\{\frac{4n+3}{3n+4}\right\}_n$ are bounded but range set of $\{n^2\}_n$ is not bounded.

(ii) Note that $\mathbb{N}$ is unbounded above, as there is no real $\lambda \in \mathbb{R}$ for which $n \leq \lambda \quad \forall n \in \mathbb{N}$.

So an interesting question is that when n becomes arbitrary large without any bound, then what will be the fate of $\{a_n\}_n$?

Consider the above examples : As n becomes larger and larger, $\frac{1}{n}$ becomes smaller & smaller we say that, the difference between $\frac{1}{n}$ and 0 decreases steadily. Neither $\frac{1}{n}$ coincides with zero nor it goes to the left side of 0. We say $\frac{1}{n} \to 0$ (tends to zero) as $n \to \infty$. But note that as n becomes arbitrarily large. n^2 increases more rapidly & we say that $n^2 \to \infty$ as $n \to \infty$. In case of $\{(-1)^n\}_n$, it is either +1 or -1.

Limit of a sequence in $\mathbb{R}$: A sequence $\{a_n\}_n$ is said to converge to a limit $l(\in \mathbb{R})$ if for arbitrary $\varepsilon > 0$, there exists natural number $m(\in \mathbb{N})$ such that $|a_n - l| < \varepsilon$ for all $n \geq m$.

$\lim\limits_{n\to\infty} a_n = \infty$ if for all $G > 0$ there exists $m \in \mathbb{N}$ such that $a_n > G \quad \forall n \geq m$. We say that $\{a_n\}_n$ diverges to ∞.

To explain this definition, we take $a_n = \frac{1}{n}$ as mentioned earlier. We have seen that $\frac{1}{n} \to 0$ as $n \to \infty$.

let $\varepsilon = \frac{7}{1000}$. Then $\left|\frac{1}{n} - 0\right| < \frac{7}{1000}$, if $n > \frac{1000}{7}\left(= 142\frac{6}{7}\right)$

so $m = 143$ & for this $\left|\frac{1}{n} - 0\right| < \frac{7}{1000}$, $n \geq 143$

Let us change $\varepsilon = \frac{8}{3439}$. Then $\left|\frac{1}{n} - 0\right| < \frac{8}{3439}$ if $n > \frac{3439}{8}\left(= 429\frac{7}{8}\right)$

So $m = 430$ & then $\left|\frac{1}{n} - 0\right| < \frac{8}{3439}$ if $n \geq 430$

These two simple examples exhibit the dependence of m on the arbitrary positive value of ε.

We state the following results without proof at this stage :

(a) A Convergent sequence in $\mathbb{R}$ is necessarily bounded but a bounded sequence may not be convergent $\left(Ex.\left\{(-1)^n\right\}_n\right)$.

(b) Limit of a sequence, if exists, is unique.

(c) Cauchy's general principle of convergence : A necessary & sufficient condition for the convergence of $\{a_n\}_n$ is that given $\varepsilon > 0$, there exists natural number $m(\in \mathbb{N})$ such that $|a_{n+p} - a_n| < \varepsilon \ \forall\, n \geq m, \quad p \in \mathbb{N}$.

(d) Sandwich rule : Let $a_n < b_n < c_n$ for all $n \geq m$ (or for all n) and $\{a_n\}_n, \{c_n\}_n$ both converge to same limit $l(\in \mathbb{R})$. Then $\lim\limits_{n\to\infty} b_n$ exists & = l.

(iii) Monotonic sequences in $\mathbb{R}$

A sequence $\{a_n\}_n$ in $\mathbb{R}$ is said to be monotonic increasing if $a_{n+1} \geq a_n$ for all n, but if $a_{n+1} \leq a_n$ for all n, $\{a_n\}_n$ is said to be monotonic decreasing sequence in $\mathbb{R}$.

We state the following results without proof :

(a) A monotonic increasing sequence $\{a_n\}_n$ in $\mathbb{R}$ is convergent if and only if $\{a_n\}_n$ is bounded above and $a_n \to \sup a_n$. If $\{a_n\}_n$ be unbounded above, then $\lim\limits_{n\to\infty} a_n = \infty$ (diverges to ∞)

(b) A monotonic decreasing sequence in $\mathbb{R}$ is convergent if and only if $\{a_n\}_n$ is bounded below and and $a_n \to \inf\limits_n a_n$. If $\{a_n\}_n$ be unbounded below, then $\lim\limits_{n\to\infty} a_n = -\infty$ (diverges to $-\infty$)

(iv) The following results are easily deducible following definition and basic results :

If $\lim\limits_{n\to\infty} a_n = l(\in \mathbb{R})$, $\lim\limits_{n\to\infty} b_n = m(\in \mathbb{R})$, then

$\lim\limits_{n\to\infty}(a_n \pm b_n), = l \pm m$, $\lim\limits_{n\to\infty}(a_n b_n) = lm$,

$\lim\limits_{n\to\infty}\left(\dfrac{a_n}{b_n}\right) = \dfrac{l}{m}$ provided $b_n \neq 0 \ \forall n$ and $m \neq 0$.

(c) Accumulation point (or limit point) of a set

Let $S(\subset \mathbb{R})$ be a set and $\xi \in \mathbb{R}$. ξ is said to be an accumulation point (or limit point) of S if there exists a sequence of distinct elements $\{x_n\}_n$ of S such that $x_n \to \xi$ as $n \to \infty$. '0' is limit point of $S = \left\{\dfrac{1}{n} : n \in \mathbb{N}\right\}$. 1 is limit point of $T\left\{1+\dfrac{1}{n} : n \in \mathbb{N}\right\}$ etc. Note that $0 \notin S, 1 \notin T$.

Note that a finite set has no accumulation point. The set $U = \{n^2; n \in \mathbb{N}\}$ has no accumulation point in $\mathbb{R}$.

(D) Neighbourhood of a point & Interior point of a set :

(i) Let $x \in \mathbb{R}$. By *a* δ-neighbourhood of *x*, we mean the interval $(x-\delta, x+\delta)$ where $\delta > 0$. This is denoted by $N(x,\delta)$ or $N_\delta(x)$.

The set $N(x,\delta)-\{x\}$ is called the deleted δ-neighbourhood $(\text{or } \delta - nbd)$ of *x*, denoted by $N'(x,\delta)$ or $N'_\delta(x)$. $\mathrm{U}(\subset \mathbb{R})$ is nbd of $x \in \mathbb{R}$ if $\exists$ an open interval I such that $x \in I \subset U$ for example, $\left(1-\frac{1}{n}, 1+\frac{1}{n}\right)$ is a neighbourhood of 1. The set $\mathbb{R}$ (of all real numbers) is a neighbourhood of each of its points. The situation is different in case of Q, the set of rational numbers for if $\xi \in Q$, then every $(\xi-\delta,\ \xi+\delta)$ contains rational as well as irrational points also. So Q is not a neighbourhood of its points.

(ii) Let $D \subset \mathbb{R}$. We say that $x \in D$ is interior point of D if there exists a neighbourhood of *x*, say $(x-\delta,\ x+\delta)$, which is contained in *D*.

For example consider $[a,b]=\{x : a \le x \le b\}$

Let $a<c<b$. we take $0<\delta<\min\{c-a, b-c\}$ & so $(c-\delta, c+\delta) \subset (a,b)$, so *c* is interior point of the set but *a*, *b* are not interior points of it.

Accumulation point can also be defined as follows :

Let $S \subset \mathbb{R}$ and $\xi \in \mathbb{R}$. If every deleted neighbourhood of ξ, $N'(\xi,\delta) \cap S \ne \phi$, then ξ is accumulation point of *S*.

This can be shown that $N'(\xi,\delta) \cap S$ is an infinite set. On the basis of this approach, it obviously follows that a finite set $(\subset \mathbb{R})$ has no accumulation point.

On the basis of these pre-requisites, we are now in a position to introduce the concept of limit of a function.

1.4 Limit of function

Let $f: D(\subset \mathbb{R}) \to \mathbb{R}$ and p be an accumulation point of D.

(A) Sequential approach : $\lim\limits_{x \to p} f(x) = l(\in \mathbb{R})$ if for every sequence $\{x_n\}_n$, $x_n \in D$ for all n, $x_i \neq x_j$ if $i \neq j$, $x_n \neq p$, converging to p, the sequences $\{f(x_n)\}_n$ converge to same limit $l(\in \mathbb{R})$.

If on the otherhand, $\{f(x_n)\}_n$ converge to different limits for different $\{x_n\}_n$'s we say that the limit does not exist.

To explain the matter, let us consider the following examples :

Example :

(i) $\lim\limits_{x \to 0} \sin\frac{1}{x}$: Note that the sequences $\left\{\frac{2}{2n\pi}\right\}_n$ and $\left\{\frac{2}{(2n+1)\pi}\right\}_n$ both converge to zero. But $\{\sin n\pi\}_n$ converges to zero whereas $\left\{\sin\left(n\pi + \frac{\pi}{2}\right)\right\}_n$ is not convergent, (n even and n odd give different limits). So by above definition, $\lim\limits_{x \to 0} \sin\frac{1}{x}$ does not exist.

(2) $\lim\limits_{x \to 0} \frac{1}{x} \sin\frac{1}{x}$

For $x_n = 1/n\pi (\to 0)$, $\frac{1}{x_n}\sin\frac{1}{x_n} \to 0$ but for $y_n = \frac{1}{\left(2n+\frac{1}{2}\right)\pi} \to 0$

$$\lim_{n \to \infty} f(y_n) = \lim_{n \to \infty}\left(2n+\frac{1}{2}\right)\pi \ \sin\left(2n+\frac{1}{2}\right)\pi = \infty$$

So $\lim_{x\to 0}\frac{1}{x}\sin\frac{1}{x}$ does not exist.

(B) ($\varepsilon-\delta$ approach) let $\varepsilon>0$ be any number. If corresponding to such ε, there exists $\delta>0$ such that $|f(x)-l|<\varepsilon$ whenever $x\in \mathbb{N}'(p,\delta)\cap D$. we say that $\lim_{x\to p} f(x)$ exists and $=l(\in\mathbb{R})$.

Here $x\in N'(p,\ \delta)\cap D$ can be written as $0<|x-p|<\delta$ or $p-\delta<x<p$, $p<x<p+\delta$, $x\in D$.

(C) The two definitions stated in (A) and (B) are equivalent :

Proof : Let $\lim_{x\to p} f(x)=l(\in\mathbb{R})$ in the sense of $\varepsilon-\delta$ definition.

Then for arbitrary $\varepsilon>0$, there exists $\delta>0$ such that

$$|f(x)-l|<\varepsilon \text{ wherever } 0<|x-p|<\delta \quad \text{(i)}$$

As p is accumulation point of D, so there exists a sequence

$\{x_n\}_n\ (x_n\in D\ \forall n,\ x_i\neq x_j \text{ if } i\neq j,\ x_n\neq p \text{ for all } n)$ which converges to p. Hence corresponding to above $\delta>0$, there exists natural number m such that

$0<|x_n-p|<\delta$ for all $n\geq m$ (2)

Combining (1) & (2), $|f(x_n)-l|<\varepsilon$ for all $n\geq m$

Note that m depends on ε (as m depends on δ & δ depends on ε).

So $\lim_{n\to\infty} f(x_n)=l(\in\mathbb{R})$ and $\{f(x_n)\}_n$ converges to $l(\in\mathbb{R})$.

Next let $\lim_{x\to p} f(x)=l(\in\mathbb{R})$ following sequential criterion.

If possible let $\lim_{x \to p} f(x) = l$ does not hold in the sense of $\varepsilon - \delta$ definition.

Then for some number $\varepsilon > 0$, the corresponding δ does not exist. That indicates, however small $\delta > 0$ may be, there exists always at least $x'(\neq p)$ for which $0 < |x' - p| < \delta$ nonetheless $|f(x') - l| \geq \varepsilon$.

Let us consider a decreasing positive termed sequence $\{\delta_n\}_n$ converging to zero (in particular, $\delta_n = \frac{1}{n}$ for all $n \in \mathbb{N}$). Then for every δ_n, x'_n can be found such that $0 < |x'_n - p| < \delta_n$ nonetheless $|f(x'_n) - l| \geq \varepsilon$. $\delta_n \to 0 \Rightarrow x'_n \to p$ by Sandwich rule. By assumption, $\{f(x'_n)\}_n$ converges to l. But $|f(x'_n) - l| \geq \varepsilon$.

Thus we arrive at a contradiction. So $\varepsilon - \delta$ definition follows from that of sequential approach. Thus the two definitions are equivalent.

(D) One sided limits

(i) Let p be an accumulation point of D from the left (i.e. $x_n \to p, x_n < p \ \forall n, x_n \in D$ etc) or f has been defined in some left-deleted neighbourhood of p. If for arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - l| < \varepsilon$ whenever $p - \delta < x < p$, we say that $\lim_{x \to p-} f(x)$ $\left(\text{or } \lim_{x \to p-0} f(x)\right)$ exists and $= l_1 (\in \mathbb{R})$. This is commonly known as left hand limit of $f(x)$ as $x \to p$.

(ii) Let p be an accumulation point of D from the right (i.e $x_n \to p,\ x_n > p\ \forall n$, $x_n \in D$ etc.) or f has been defined in some right deleted neighbourhood of p. If for arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - l_2| < \varepsilon$ whenever $p < x < p + \delta$, we say that $\lim_{x \to p+} f(x)$ $\left(\text{or } \lim_{x \to p+0} f(x)\right)$ exists and $= l_2 (\in \mathbb{R})$. This is commonly known as right hand limit of $f(x)$ as $x \to p$.

(E) In this connection, the following result is useful in determining the existence of limit. Let $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}$ and let p be (both sided) accumulation point of D (or f has been defined in both sided deleted neighbourhood of p).

Then $\lim_{x \to p} f(x) = l(\in \mathbb{R})$ if and only if $\lim_{x \to p-0} f(x) = \lim_{x \to p+0} f(x) = l$

Proof : let $\lim_{x \to p} f(x) = l(\in \mathbb{R})$

Following $\varepsilon - \delta$ definition, corresponding to arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - l| < \varepsilon$ whenever $0 < |x - p| < \delta, x \in D$

$\Rightarrow |f(x) - l| < \varepsilon$ whenever $p - \delta < x < p$ as well as $p < x < p + \delta$.

$\Rightarrow \lim_{x \to p-} f(x) = l = \lim_{x \to p+} f(x)$

Converse let $\lim_{x \to p-} f(x) = l = \lim_{x \to p+} f(x)$

Let $\varepsilon > 0$ be any number. Corresponding to ε, there exists $\delta_1 > 0, \delta_2 > 0$ such that $|f(x) - l| < \varepsilon$ whenever $p - \delta_1 < x < p$ & $|f(x) - l| < \varepsilon$ whenever $p < x < p + \delta_2$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then for $0 < |x - p| < \delta$, $|f(x) - l| < \varepsilon \Rightarrow \lim_{x \to p} f(x) = l$

Examples (i) $f(x) = \begin{cases} 3x + 7, & x < 1 \\ 2x + 11, & x > 1 \end{cases}$

Here $\lim_{x \to 1-} f(x) = 10$, $\lim_{x \to 1+} f(x) = 13$ & so $\lim_{x \to 1} f(x)$ does not exist.

(2) $f(x) = \begin{cases} 7x + 3, & x < 2 \\ 8x + 1, & x > 2 \end{cases}$

Here $\lim_{x\to 2-} f(x) = 17,\ \lim_{x\to 2+} f(x) = 17$

Let $\varepsilon > 0$ be any number. Corresponding to ε, there exists $\delta_1 > 0, \delta_2 > 0$ such that $|7x+3-17| < \varepsilon$ i.e. $|x-2| < \frac{\varepsilon}{7}$ whenever $2-\delta_1 < x < 2$ & so $\delta_1 = \frac{\varepsilon}{7}$ is admissible & $|8x+1-17| < \varepsilon$ i.e. $|x-2| < \frac{\varepsilon}{8}$ whenever $2 < x < 2+\delta_2$ & so $\delta_2 = \frac{\varepsilon}{8}$ is admissible. Taking $\delta = \min\{\delta_1, \delta_2\}$, we get $\lim_{x\to 2} f(x) = 17$

(F) Cauchy Criterion for the existence of limit

Let $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}$ and p be an accumulation point of D.

A necessary and sufficient condition for the existence of $\lim_{x\to p} f(x)$ is that given $\varepsilon > 0$, there exists a deleted neighbourhood of $p, \mathbb{N}'(p,\delta)$ such that

$|f(x) - f(y)| < \varepsilon$ whenever $x, y \in \mathbb{N}'(p,\delta) \cap D$

Proof : Let $\lim_{x\to p} f(x) = l\,(\in \mathbb{R})$

Let $\varepsilon > 0$ be any number. Corresponding to ε, there exists a deleted neighbouhood $\mathbb{N}'(p,\delta)$ such that $|f(x) - l| < \varepsilon/2$ whenever $x \in \mathbb{N}'(p,\delta) \cap D$

If moreover $y \in \mathbb{N}'(p,\delta) \cap D, |f(y) - l| < \varepsilon/2$ As a result,

$|f(x) - f(y)| \le |f(x) - l| + |f(y) - l| < \varepsilon$ holds.

Converse : Let for given $\varepsilon > 0$, there exists a deleted neighbourhood $\mathbb{N}'(p,\delta)$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in \mathbb{N}'(p,\delta) \cap D$

Let p be accumulation point of D. So there exists $\{x_n\}_n \left(x_n \in D\ \forall_n, x_i \neq x_j\right.$ if $\left. i \neq j,\ x_n \neq p\right)$ which converges to p. Hence corresponding to above $\delta(>0)$, there exists $m \in \mathbb{N}$ such that $x_n \in \mathbb{N}'(p, \delta) \cap D$ for all $n \geq m$.

Therefore, $|f(x_n) - f(x_k)| < \varepsilon$ for all $n,\ k \geq m$.

So by Cauchy's general principle of convergence of a sequence, $\{f(x_n)\}_n$ is convergent and so $\lim_{x \to p} f(x)$ exists.

Illustration : Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1, \text{if } x \text{ is rational} \\ -1, \text{if } x \text{ is irrational} \end{cases}$

Let $a \in (0, 1)$. Note that for any $\delta > 0, N'(a, \delta) \cap (0, 1)$ contains both rational as well as irrational points. If such rational be x & such irrational be y, then $|f(x) - f(y)| = |1 - (-1)| = 2 \not< $ arbitrary $\varepsilon > 0$.

So by Cauchy Criterion, $\lim_{x \to a} f(x)$ does not exist.

(G) Infinite limits and Limit at infinity

(i) Infinite limits :

Let $f : D \to \mathbb{R}$ and p be an accumulation point of $D(\subset \mathbb{R})$. Then $f(x)$ is said to be tend to ∞ as $x \to p$, if given any $G > 0$ (as large as we please), there exists $\delta > 0$ such that

$f(x) > G$ whenever $x \in N'(p, \delta) \cap D$.

If we opt for sequential approach, if for $\{x_n\}_n \left(x_n \in D\ \forall n, x_i \neq x_j \text{ if } i \neq j, x_n \neq p\right)$ converges to p, $\{f(x_n)\}_n$ diverges to ∞, we say that $\lim_{x \to p} f(x) = \infty$

Illustration : $\lim_{x \to 0^+} \frac{1}{x} = \infty$

For any $G > 0, \frac{1}{x} > G$ if $x < \frac{1}{G} (\to 0 \text{ as } G \to \infty)$.

If for given $G > 0$ (as large as we please), there exists $\delta > 0$ such that $f(x) < -G$ whenever $x \in N'(p, \delta) \cap D$, we say that $\lim_{x \to p} f(x) = -\infty$

(ii) Limit at infinity

Let $f : D \to \mathbb{R}$ where D is unbounded above.

If for given $\varepsilon > 0$, there exists $G > 0$ such that

$|f(x) - l| < \varepsilon$ whenever $x \in (G, \infty) \cap D$,

We say that $\lim_{x \to \infty} f(x) = l (\in \mathbb{R})$ $ex \cdot \lim_{x \to \infty} \frac{1}{x} = 0$.

Next let $f : D \to \mathbb{R}$ where D is unbounded below.

If for given $\varepsilon > 0$, there exists $G > 0$ such that

$|f(x) - l| < \varepsilon$ whenever $x \in (-\infty, G)$, we say that $\lim_{x \to -\infty} f(x) = l (\in \mathbb{R})$

Illustration (1) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e, x \in \mathbb{R}$

To solve this, we will assume the very standard limit of sequence $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e (\in \mathbb{R})$.

We can take $x > 1$. There exists natural number n such that

$n \leq x < n + 1$

$$\Rightarrow \left(1 + \frac{1}{n}\right)^{n+1} > \left(1 + \frac{1}{x}\right)^x > \left(1 + \frac{1}{n+1}\right)^n$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left\{\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right)\right\} = e \text{ and}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{1+\frac{1}{n+1}} = e$$

So, $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$

(2) $\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x} = e$

We take $x=-y$ and So $y\to\infty$ and $x\to-\infty$

$$\left(1+\frac{1}{x}\right)^{x} = \left(1-\frac{1}{y}\right)^{-y} = \left(\frac{y}{y-1}\right)^{y} = \left(1+\frac{1}{y-1}\right)^{y-1}\cdot\left(1+\frac{1}{y-1}\right)\to e \text{ as } y\to\infty.$$

In this connection, we state the following result :

Let $f:(a,\infty)\to\mathbb{R}$, Then $\lim_{x\to\infty} f(x)$ exists if and only if for every $\varepsilon>0$, there exists $X(>a)$ such that $|f(x)-f(y)|<\varepsilon\ \forall\ x,y>X$.

(iii) Infinite limits at infinity

Let $f:D\to\mathbb{R}$ where $D(\subset\mathbb{R})$ is unbounded above.

Let $G>0$ be any number, as large as we please.

Corresponding to G, there exists $K(\in\mathbb{R})$ such that $f(x)>G$ for all $x>K$, we say that $\lim_{x\to\infty} f(x)=\infty$.

Let D be unbounded below, if corresponding to $G>0$ (as large as we please), there exists $K(\in\mathbb{R})$ such that $f(x)>G$ for all $x<K$, we say that $\lim_{x\to-\infty} f(x)=\infty$.

But if $f(x)<-G$ for all $x<K$, we say $\lim_{x\to-\infty} f(x)=-\infty$.

Example : $\lim_{x\to\infty} \log_a x = \infty,\ a > 1$

Let $G > 0$ be any arbitrary number. If we take $a^G = M$, then

$x > M \Rightarrow \log_a x > G$. Hence $\lim_{x\to\infty} \log_a x = \infty$.

(H) Some standard limits :

(i) $\lim_{x\to 0} \frac{\sin x}{x} = 1$

(ii) $\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a e$ where $a > 0, a \neq 1$

(iii) $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a, a > 0$

(iv) $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}, a > 0$

(I) Algebra of limits :

Let $g, f : D \to \mathbb{R}$ when $D \subset \mathbb{R}$ and p be an accumulation point of D.

Let $\lim_{x\to p} f(x) = l(\in \mathbb{R}),\ \lim_{x\to p} g(x) = m(\in \mathbb{R})$.

Then (i) $\lim_{x\to p} \{f(x) \pm g(x)\} = l \pm m$

(ii) $\lim_{x\to p} \{f(x)g(x)\} = lm$

(iii) $\lim_{x\to p} \frac{f(x)}{g(x)} = \frac{l}{m}$ where $g(x) \neq 0$ and $m \neq 0$.

Proof : (i) Let $\varepsilon > 0$ be any number. Corresponding to ε, there exists $\delta_1 > 0,\ \delta_2 > 0$ such that $|f(x) - l| < \frac{\varepsilon}{2}$ whenever $0 < |x - p| < \delta_1,\ x \in D$ and $|g(x) - m| < \frac{\varepsilon}{2}$ whenever

$0<|x-p|<\delta_2, x \in D$.

Let $\delta=\min\{\delta_1, \delta_2\}$. So for $0<|x-p|<\delta$, $x \in D$ both hold.

Hence $|\{f(x) \pm g(x)\}-\{l \pm m\}| \leq |f(x)-l|+|g(x)-m|<\varepsilon$

whenever $0<|x-p|<\delta$, $x \in D$

$\Rightarrow \lim_{x \to p}\{f(x) \pm g(x)\}=l \pm m=\lim_{x \to p} f(x) \pm \lim_{x \to p} g(x)$

Note : (1) This result can be generalised for finite number of functions.

(2) The converse of the result is not true, in general

Let $f(x)=\begin{cases}1, \text{ if } x \text{ is rational} \\ 0, \text{ if } x \text{ is irrational}\end{cases}$ and $g(x)=\begin{cases}0, \text{ if } x \text{ is rational} \\ 1, \text{ if } x \text{ is irrational}\end{cases}$

Let $p \in \mathbb{R}$. Every deleted *nbd* of *p* contains both rational (say *a*) and irrational *b* (say) points. Then in case of both *f* and *g*, $|f(a)-f(b)|$ or $|g(a)-g(b)|=1 \nless$ arb ε.

So neither $\lim_{x \to p} f(x)$ nor $\lim_{x \to p} g(x)$ exists. But $f(x)+g(x)=1$ and

$\lim_{x \to p}\{f(x)+g(x)\}=1$.

(ii) To establish it we will first show that as $\lim_{x \to p} g(x)$ exists, so there exists a deleted neighbourhood of *p*, in which *g* is bounded.

There exists $\delta_1>0$ such that $|g(x)-m|<1$ where $0<|x-p|<\delta_1$, $x \in D$ (or $x \in N'(p, \delta) \cap D$)

$\Rightarrow |g(x)|<1+|m|$ in $N'(p, \delta_1) \cap D$

$\Rightarrow g$ is bounded in $N'(p, \delta_1) \cap D$

$$|f(x)g(x)-lm| = |g(x)\{f(x)-l\}+l(g(x)-m)| \le |g(x)||f(x)-l|+|l||g(x)-m|$$
.....(1)

As $\lim_{x\to p} g(x)$ exists, so there exists $\delta_1 > 0$ such that $|g(x)| < \lambda$ for some $\lambda \in \mathbb{R}^+$ in $N'(p, \delta_1)\cap D$ .....(1)

Let $\varepsilon > 0$ be any number, corresponding to ε, there exists $\delta_2 > 0, \delta_3 > 0$ such that

$|f(x)-l| < \dfrac{\varepsilon}{2\lambda}$ whenever $x \in N'(p, \delta_2)\cap D$ ....(2)

and $|g(x)-m| < \dfrac{\varepsilon}{2(|l|+1)}$ whenever $x \in N'(p, \delta_3)\cap D$ ....(3)

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then in $N'(p, \delta)\cap D$, by (1), (2), (3)

$$|f(x)g(x)-lm| < \lambda \cdot \frac{\varepsilon}{2\lambda} + |l| \cdot \frac{\varepsilon}{2(|l|+1)}$$

$$\Rightarrow |f(x)g(x)-lm| < \varepsilon \text{ in } N'(p, \delta)\cap D$$

$$\Rightarrow \lim_{x\to p} f(x)g(x) = lm = \left(\lim_{x\to p} f(x)\right)\left(\lim_{x\to p} f(g)\right)$$

Note : (1) This result can be generalised for finite number of functions.

(2) $\lim_{x\to 0} \sin\dfrac{1}{x}$ does not exist but $\lim_{x\to 0} x\sin\dfrac{1}{x} = 0$

Let $\varepsilon > 0$ be any number $\left|x\sin\dfrac{1}{x} - 0\right| \le |x| < \varepsilon$ whenever $x \in N'(0, \delta)\cap D_f$ where $\delta \equiv \delta(\varepsilon)$.

(3) If $g(x)$ be bounded on D and $\lim_{x\to p} f(x) = 0$, then $\lim_{x\to p} f(x)g(x)$ exists $= 0$.

(iii) $\left|\frac{f(x)}{g(x)}-\frac{l}{m}\right|=\left|\frac{m\{f(x)-l\}-l(g(x)-m)}{m\,g(x)}\right|$

$$\leq \frac{|m|\,|f(x)-l|+|l|\,|g(x)-m|}{|m|\,|g(x)|} \quad \text{.... (1)}$$

As $\lim\limits_{x\to p} g(x)=m(\neq 0)$, there exists $\delta_1>0$ such that

$|g(x)-m|<\frac{|m|}{2}$ whenever $x\in N'(p,\ \delta_1)\cap D$ .... (2)

$\Rightarrow |g(x)|>\frac{|m|}{2}$ whenever $x\in N'(p,\ \delta_1)\cap D$

Let $\varepsilon>0$ be any number.

As $\lim\limits_{x\to p} f(x)=l$, corresponding to ε, there exits $\delta_2>0$ such that

$|f(x)-l|<\frac{\varepsilon|m|}{4}$ whenever $x\in N'(p,\ \delta_2)\cap D$ ... (3)

As $\lim\limits_{x\to p} g(x)=m$, corresponding to ε, there exists $\delta_3>0$ such that

$|g(x)-m|<\frac{\varepsilon|m|^2}{4(|l|+1)}$ whenever $x\in N'(p,\ \delta_3)\cap D$ ... (4)

Let $\delta=\min\{\delta_1,\ \delta_2,\ \delta_3\}$. So whenever $x\in N'(p,\ \delta)\cap D$, (2), (3) (4) hold.

Recalling 1. $\left|\frac{f(x)}{g(x)}-\frac{l}{m}\right|<\frac{2}{|m|^2}\left|\left\{\frac{\varepsilon.|m|^2}{4}+\frac{|l|.\varepsilon|m|^2}{4(|l|+1)}\right\}\right|$

$$\Rightarrow \left|\frac{f(x)}{g(x)}-\frac{l}{m}\right|<\varepsilon \text{ whenever } x\in N'(p,\delta)\cap D$$

$$\Rightarrow \lim_{x\to p}\frac{f(x)}{g(x)}=\frac{l}{m}=\frac{\lim\limits_{x\to p} f(x)}{\lim\limits_{x\to p} g(x)}$$

Note : Neither $\lim\limits_{x\to 0}\frac{1}{x}$ nor $\lim\limits_{x\to 0}\sin\frac{1}{x}$ exists, but $\lim\limits_{x\to 0} x\sin\frac{1}{x}$ exists & = 0.

So the Converse of (iii) is not, in general, true.

Illustration : Evaluate (1) $\lim\limits_{x\to 0}\frac{(e^x-1)\tan^2 x}{x^3}$ (2) $\lim\limits_{x\to 0}\frac{\tan x-\sin x}{x^3}$

(3) $\lim\limits_{x\to\frac{\pi}{6}}\frac{\sin\left(x-\frac{\pi}{6}\right)}{\left(\sqrt{3}-2\cos x\right)}$

(1) $=\lim\limits_{x\to 0}\left\{\frac{e^x-1}{x}.\left(\frac{\sin x}{x}\right)^2.\left(\frac{1}{\cos x}\right)^2\right\}$

$$=\lim_{x\to 0}\frac{e^x-1}{x}.\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2.\lim_{x\to 0}\left(\frac{1}{\cos^2 x}\right)=1 \text{ (As all exist)}$$

So limit is 1.

(2) $\lim\limits_{x\to 0}\frac{\sin x(1-\cos x)}{x^3\cos x}=\lim\limits_{x\to 0}\left\{\frac{\sin x}{x}.\frac{1}{\cos x}.\frac{2\sin^2\frac{x}{2}}{x^2}\right\}$

$$= \lim_{x\to 0}\left\{\frac{\sin x}{x}\cdot\frac{1}{\cos x}\cdot\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)\right\}\frac{1}{2} = 1.1.1.\frac{1}{2} = \frac{1}{2}$$

(3) (Method of substitution) Put $x - \frac{\pi}{6} = t$ and so $x \to \frac{\pi}{6} \Leftrightarrow t \to 0$.

$$\text{Given limit} = \lim_{t\to 0}\frac{\sin t}{\sqrt{3} - 2\cos\left(t + \frac{\pi}{6}\right)} = \lim_{t\to 0}\frac{\sin t}{\sqrt{3} + \sqrt{3}\cos t + \sin t}$$

$$= \lim_{t\to 0}\frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{\sqrt{3}\left(2\sin^2\frac{t}{2}\right) + 2\sin\frac{t}{2}\cos\frac{t}{2}} = \lim_{t\to 0}\frac{\cos\frac{t}{2}}{\sqrt{3}\sin\frac{t}{2} + \cos\frac{t}{2}} = 1$$

(J) Neighbourhood properties :

(a) Let $f : D \to \mathbb{R}, D \subset \mathbb{R}$ and p be an accumulation point of D. Let $\lim_{x\to p} f(x) = l (\in \mathbb{R})$. Then

(i) f is bounded in some deleted *nbd* of p

(ii) If l be greater than some real number K, then there exists a deleted *nbd* of p in which $f(x) > K$.

(iii) If l be less than some real number μ, then there exists a deleted *nbd* of p in which $f(x) < \mu$.

Proof : (i) Proved earlier in I(ii)

(ii) Let $0 < \varepsilon < l - K$. Corresponding to this ε, then exists $\delta > 0$ such that

$|f(x) - l| < \varepsilon$ for all $x \in N'(p, \delta) \cap D$

$\Rightarrow l-\varepsilon < f(x) < l+\varepsilon \forall x \in N'(p,\delta)\cap D$

Considering the above choice of $\varepsilon, f(x) > K$ in $N'(p,\delta)\cap D$

(iii) As in (ii), taking $0 < \varepsilon < \mu - l$.

(b) Let $f, g : D \to \mathbb{R}$ where $D(\subset \mathbb{R})$ & p be an accumulation point of D.

Let $\lim\limits_{x\to p} f(x) = A(\in \mathbb{R}), \lim\limits_{x\to p} g(x) = B(\in \mathbb{R})$.

If $A < B$, then there exists a deleted neighbourhood of p in which $f(x) < g(x)$.

Proof : Let $A < C < B$.

As $\lim\limits_{x\to p} f(x) = A$, there exists $\delta_1 > 0$ such that $|f(x) - A| < C - A$ for all $x \in N'(p,\delta_1)\cap D$.

As $\lim\limits_{x\to p} g(x) = B$, there exists $\delta_2 > 0$ such that $|g(x) - B| < B - C$ for all $x \in N'(p,\delta_2)\cap D$.

Let $\delta = \min\{\delta_1, \delta_2\}$. So in $N'(p,\delta)\cap D$, both hold.

In $N'(p,\delta)\cap D,\ f(x) < C - A + A = C = B - (B - C) < g(x)$ holds.

(c) Sandwich property :

Let $f, g, h : D \to \mathbb{R}$ where $D(\subset \mathbb{R})$. Let $f(x) < g(x) < h(x)$ for all $x \in D$ & let p be an accumulation point of D. Given that $\lim\limits_{x\to p} f(x) = l,\ \lim\limits_{x\to p} h(x) = l(l \in \mathbb{R})$.

Then $\lim\limits_{x\to p} g(x) = l$.

Proof : Let $\varepsilon > 0$ be any number. Corresponding to this ε, there exists $\delta_1 > 0, \delta_2 > 0$ such that $|f(x) - l| < \varepsilon$ in $N'(p,\delta_1)\cap D$ & $|h(x) - l| < \varepsilon$ in $N'(p,\delta_2)\cap D$.

Let $\delta = \min\{\delta_1, \delta_2\}$. So in $N'(p,\delta)\cap D$,

$l-\varepsilon < f(x) < g(x) < h(x) < l+\varepsilon \Rightarrow |g(x)-l| < \varepsilon$ in $N'(p,\delta)\cap D$

So $\lim\limits_{x\to p} g(x) = l$.

(d) $f, g : D \to \mathbb{R}$ where $D(\subset \mathbb{R})$, p be accumulation point of D and

let $\lim\limits_{x\to p} f(x) = l(\in \mathbb{R})$, $\lim\limits_{x\to p} g(x) = m(\in \mathbb{R})$. If $f(x) < g(x)$ in D, then $l \le m$.

Proof : If possible let $l > m$ & let $o < \varepsilon < \frac{l-m}{10}$, corresponding to such ε, there exists $\delta_1, \delta_2 >$ such that $|f(x)-1| < \varepsilon$ in $N'(p,\delta_1)\cap D$ & $|g(x)-m| < \varepsilon$ in $N'(p,\delta_2)\cap D$.

If $\delta = \min\{\delta_1, \delta_2\}$, then in $N'(p,\delta)\cap D$, both hold.

In $N'(p,\delta)\cap D$, $l-\varepsilon < f(x) < g(x) < m+\varepsilon \Rightarrow l-m < 2\varepsilon \Rightarrow 10\varepsilon < 2\varepsilon$ — absurd as $\varepsilon > 0$.

So $l \le m$.

[You can take $f(x) = 1-x$, $g(x) = 1+x$ where $x > 0$. $f(x) < g(x)$ for all x and $\lim\limits_{x\to 0+} f(x) = 1 = \lim\limits_{x\to 0+} g(x)$.]

K. Infinitesimal :

(a) $f : D \to \mathbb{R}(D \subset \mathbb{R})$ is said to be infinitesimal as $x \to a$ if $\lim\limits_{x\to a} f(x) = 0$.

(b) If $f, g : D \to \mathbb{R}$ are infinitesimals, then $f \pm g$, fg are also so.

(c) If $f : D \to \mathbb{R}$ be infinitesimal as $x \to a$ and $g : D \to \mathbb{R}$ be bounded, then fg is infinitesimal.

(d) We say $f = o(g)$ (or f is of little –*oh* of g over D) if

$f(x)=\alpha(x)g(x)$ where $\alpha(x)$ is infinitesimal.

(e) We say $f=O(g)$ (or *f* is of big *–oh* of *g* over *D*) if $f(x)=\beta(x)g(x)$ where $\beta(x)$ is bounded on *D*.

(f) The functions *f* and *g* are of same order over $D(\subset\mathbb{R})$, if $f=O(g)$ and $g=O(f)$ simultaneously.

1.5 Summary

In this unit, we have defined the term functions and classified various type of functions. We have defined real valued sequences and study limit of a real sequence. We have explaind the concept of limit of functions and study some criterian for the existence of limit. We also introduced the concept of infinite limits, limit at infinity, neighbourhood properties. We have explained the Sandwich property and the concepts of infinitesimal.

1.6 Exercise

1. Find the limits (if exist)

(a) $\lim\limits_{x\to\infty}\left(\dfrac{x^3}{3x^2-4}-\dfrac{x^2}{3x+2}\right)$

(b) $\lim\limits_{x\to 0}\dfrac{\left(2x^2+|x|\right)}{x}$

(c) $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2}+3\right)$, $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2}+1\right)$ and $\lim\limits_{x\to 0}\left\{\left(\dfrac{1}{x^2}+3\right)-\left(\dfrac{1}{x^2}+1\right)\right\}$

(d) $\lim\limits_{x\to 3}\dfrac{\sqrt{(3x)}-3}{\sqrt{2x-4}-\sqrt{2}}$

(e) Apply Cauchy's principle for the existence of limit to evaluate $\lim\limits_{x\to 0}\dfrac{1+x}{1-x}$.

2. Choose the correct one : $\lim_{x\to 0} \frac{\sin[x]}{[x]}$

(a) the limit exists and is 1

(b) the limit does not exist.

(c) if at $x = 0,\ f(0) = 0$, the limit will exist

(d) if at $x = 0,\ f(0) = 1$, the limit will exist.

Unit-2 □ Continuity of Functions

Structure

2.0 Objectives

This unit gives

- The concept of continuity of a real fuction
- Classification of discontinuity
- Neighbourhood properties of a continous function
- The behaviour of continuous function in a closed and bounded interval
- The concept of uniform continuity

2.1 Introduction

A general function from $\mathbb{R}$ to $\mathbb{R}$ can be very convoluted indeed, which means that we will not be able to make many meaningful statements about general functions. To develop a useful theory, we must instead restrict the class of functions we consider. Intuitively we require that the functions be sufficiently 'nice', and see what properties we can deduce from such restrictions. The study of continuous functions is a case in point by requiring a function to be continuous, we obtain enough information to deduce powerful theorems, such as the Intermediate value theorem. However, the definition of continuity is flexible enough that there are a wide, and interesting, variety of continuous functions. Indeed, many functions that come up in real-world problems are continuous, which makes the definition pleasing from both an aesthetic and practical point of view.

2.2 Definition

I. (a) A function $f: D \to \mathbb{R}(D \subset \mathbb{R})$ is said to be continuous at $p \in D$ if given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$|f(x) - f(p)| < \varepsilon$ or $f(x) \in N(f(p), \varepsilon)$ whenever $x \in N(p, \delta) \cap D$.

If f is not continuous at p, then f is discontinuous at p.

(b) Let $f: D \to \mathbb{R}(D \subset \mathbb{R})$ and $p \in D$.

(i) If p is an isolated point of D (i.e. not a limit point of D), then f is continuous at p (ii) if p be limit point of D i.e. $p \in D \cap D'$ (D' is the collection of limit points of D) and if $\lim_{x \to p} f(x) = f(p)$, then f is continuous at p.

(c) Continuity in an interval $[a, b]$ or in $\{x : a \le x \le b\}$.

f is continuous in $[a, b]$ if (i) $\lim_{x \to a+0} f(x) = f(a)$ (ii) $\lim_{x \to b-0} f(x) = f(b)$ and (iii) if $a < c < b$, then $\lim_{x \to c-0} f(x) = \lim_{x \to c+0} f(x) = f(c)$.

Examples. 1. Let $f(x) = \begin{cases} x, \text{ for } x \in \left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\} \\ 1, \text{ for } x = 1 \end{cases}$

be defined on $S = \left\{1 - \frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \{1\}$.

The only accumulation point of S is 1 and all other points of S are its isolated points. Here $\lim_{x\to 1} f(x) = f(1) = 1 \Rightarrow f$ is continuous at 1, f is also continuous at the isolated points $1-\frac{1}{n} : n \in \mathbb{N}$. Hence f is continuous on S.

2. Let $f(x) = \frac{1}{(1-x)}$, $x \neq 1$. Find the points of discontinuity of $y = f\left[f\left(f(x)\right)\right]$.

$x = 1$ is a point of discontinuity of $f(x)$.

If $x \neq 1$, $f\left[f(x)\right] = \frac{1}{1-\frac{1}{(1-x)}} = \frac{x-1}{x}$, $x \neq 0 \Rightarrow x = 0$ is a point of discontinuity of $f\left[f(x)\right]$.

If $x \neq 0$, $x \neq 1$, $y = \frac{1}{1-\frac{x-1}{x}} = x$ is continuous everywhere.

So points of discontinuity of the given composite function are $x = 0$, $x = 1$.

(3) Let $E = \left\{1-\frac{1}{n} | n \in \mathbb{N}\right\} \cup [1, 2]$ and $f : E \to \mathbb{R}$ be defined by $f(x) = x^2$.

Each $1-\frac{1}{n}$ is isolated point of E and so by definition, f is continuous at all such points.

Let $p \in [1, 2]$. Then $p \in E \cap E'$ (E' derived set of E) and then $x^2 \to p^2$ or $f(x) \to f(p)$. So f is continuous at p.

Thus f is continuous on E.

(Continuation of definiton (d)) $f: D \to \mathbb{R}$ where $D(\subset \mathbb{R})$ and $p \in D \cap D'$.

f is continuous at p if for every sequence

$\{x_n\}_n (x_n \in D \ \forall n,\ x_i \neq x_j$ if $i \neq j,\ x_n \neq p)$ converging to p, $\{f(x_n)\}_n$ converges to $f(p)$.

Examples (1) Let $A = \{x \in \mathbb{R} | x > 0\}$ and let $f: A \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0, \text{ if } x \text{ is irrational} \\ \dfrac{1}{n}, \text{ if } x = \dfrac{m}{n} \text{ where } m, n \in \mathbb{N} \text{ and } (m, n) = 1 \end{cases}$$

To examine the continuity of f in A.

We require the following lemma :

Let i be any irrational number between 0 and 1.

Let p, q, n be any positive integers such that $p < q \leq n$ and n is fixed. Then there exists a neighbourhood of i which has the property that no rational number of the form $\frac{p}{q}$ belongs to it.

Proof of lemma : Let d be the least of the differences $\left|i - \frac{p}{q}\right|$ for all p, q such that $p < q \leq n$. Let δ be chosen so that $0 < \delta < d$. Then $(i - \delta,\ i + \delta)$, a nbd of i, which has the property stated above.

Let us now examine the continuity of f.

Let b be any irrational number and let $\varepsilon > 0$.

Now there exists $n_0 \in \mathbb{N}$ such that $n_0 \varepsilon > 1$ (known as Archimedean property of real numbers). By above lemma, $\delta > 0$ can be chosen so small that the nbd $(b - \delta,\ b + \delta)$ contains no rational number with denominator $< n_0$.

If then follows that for $|x-b|<\delta$, $x \in A$, we have

$$|f(x)-f(b)|=|f(x)| \leq \frac{1}{n_0}<\varepsilon \Rightarrow f \text{ is continuous at irrational point } b.$$

Let $a \in A$ be any rational point. Let $\{x_n\}_n$ be any sequence of irrational numbers in A that converges to a. Then $\lim_{n\to\infty} f(x_n)=0$ where as $f(a)>0$. Hence f is discontinuous at all rational points.

(2) (Dirichlet's function) $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases}1, \text{ if } x \text{ be rational}\\ 0, \text{ if } x \text{ be irrational}\end{cases}$$

Applying sequential approach, it can be shown that f is discontinuous everywhere.

(3) Let $f(x)=\begin{cases}x, \text{ if } x \text{ is rational}\\ 1-x, \text{ if } x \text{ is irrational}\end{cases}$

To investigate the continuity of f on $\mathbb{R}$.

Let $\varepsilon>0$ be any number.

$$\left|f(x)-f\left(\frac{1}{2}\right)\right|=\begin{cases}\left|x-\frac{1}{2}\right|, \text{ if } x \text{ is rational}\\ \left|1-x-\frac{1}{2}\right|=\left|x-\frac{1}{2}\right|, \text{ if } x \text{ is irrational}\end{cases}$$

So $\left|f(x)-f\left(\frac{1}{2}\right)\right|=\left|x-\frac{1}{2}\right|<\varepsilon$ whenever $\left|x-\frac{1}{2}\right|<\delta(=\varepsilon)$.

f is continuous at $x=\frac{1}{2}$.

Next let $x \neq \frac{1}{2}$ and x is rational. Let $\{x_n\}_n$ be a sequence of irrationals such that $\lim_{n\to\infty} x_n = x$. So $f(x_n) = 1 - x_n \to 1 - x$ as $n \to \infty$.

As $x \neq \frac{1}{2}$, so $x \neq 1 - x$ and f is discontinuous on $Q - \left\{\frac{1}{2}\right\}$.

Next let x be irrational number and let $\{y_n\}_n$ be a sequence of rational numbers such that $\lim_{n\to\infty} y_n = x$. Here $f(y_n) = y_n \to x$ as $n \to \infty$. But $f(x) = 1 - x$.

So $\lim_{n\to\infty} f(y_n) \neq f\left(\lim_{n\to\infty} y_n\right) \Rightarrow f$ is discontinuous at all irrational points.

Consequently f is continuous only at $x = \frac{1}{2}$.

Classification of discontinuities :

Let f be not continuous at $p(\in D_f)$. This discontinuity of f at p may be due to different reasons which may be classified into two types / kinds.

Definition : (a) Let f be defined in both-sided neighbourhood of point $p(\in D_f)$.

Let $\lim_{x\to p+} f(x)$ and $\lim_{x\to p-} f(x)$ both exist finitely but are unequal, then $x = p$ is known as jump discontinuity of f.

$f(p+o) - f(p-o)$ is known as height of the jump. If f has jump discontiuity on the right at a, the height of jump is $f(a+o) - f(a)$ and similarly at b, it is $f(b) - f(b-o)$, if it is left discontinuous at b.

Example : let : $[0,1] \to \mathbb{R}$ be defined by

$$f(x)=\begin{cases} 0, & \text{if } x=0 \\ \frac{1}{2}-x, & \text{if } 0<x<\frac{1}{2} \\ 1/2, & \text{if } x=\frac{1}{2} \\ \frac{3}{2}-x, & \text{if } \frac{1}{2}<x<1 \\ 1, & \text{if } x=1 \end{cases}$$

$f(0+)=\frac{1}{2}\neq f(0)$, $f\left(\frac{1}{2}-0\right)=0$, $f\left(\frac{1}{2}+0\right)=1$ so $f\left(\frac{1}{2}-0\right)\neq f\left(\frac{1}{2}+0\right)$

$f(1-0)=\frac{1}{2}\neq 1$ so, 0, $\frac{1}{2}$, 1 are points of jump discontinuity of f.

If $f(p-0), f(p+0)$ both exist and are equal but $\neq f(p)$,

then p is removable discontinuity of f $\left(\text{i.e} \lim\limits_{x\to p} f(x)\neq f(p)\right)$

Example : $f(x)=\begin{cases} 5x+7, & x<2 \\ 13, & x=2 \\ 4x+9, & x>2 \end{cases}$

$f(2-0)=17=f(2+0)$ but $f(2)=13$

$x=2$ is removable discontinuity. These two types of discontinuity are known as discontinuity of first kind or ordinary discontinuity.

(b) (i) If f is defined in both sided nbd of p including p and at least one of $f(p-0)$ & $f(p+0)$ fails to exist finitely though f is bounded in some deleted neighbourhood of p, then p is discontinutiy of second kind with finite oscillation.

Example : $f(x)=\begin{cases} \sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$

Neither $\lim_{x\to 0+} f(x)$ nor $\lim_{x\to 0-} f(x)$ exists & but f is bounded in *nbd* of 0.

(ii) f is unbounded in every nbd p and $\lim_{x\to P+0} f(x)$ or $\lim_{x\to p-0} f(x)$ is $+\infty$ or $-\infty$. Such a discontinuity is known as infinte discontinuity.

Example : $f(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ 2, & x = 0 \end{cases}$

2.3 Neighbourhood properties

Let $f : D \to \mathbb{R}$ where $D(\subset \mathbb{R})$ and p be an accumulation point of D as well as an element of D. Let f be continuous at p.

Then the following results hold :

(i) There exists a neighbourhood of p in which f is bounded.

(ii) If $f(p) \neq 0$, there exists a neighbourhood of p in which $f(x)$ & $f(p)$ have the same sign.

(iii) If in every neighbourhood of $p, f(x)$ assumes both positive & negative values, then $f(p) = 0$

The first two properties follow from the neighbourhood properties for the existence of limit.

For (iii) if $f(p) > 0$, by (ii) there exists *nbd* of p in which $f(x) > 0$ for all $x \in \mathbb{N}(p,\delta) \cap D$. But $f(x)$ have both signs in every *nbd* of p & so $f(p) \not> 0$. By similar logic, $f(p) \not< 0$. Hence $f(p) = 0$.

The converse of (iii) is not true. For example, $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Continuity of some special types of functions

(i) Let $f : D \to \mathbb{R}$ be monotone function (increasing or decreasing). Then at every point c of D, both $f(c+0)$ & $f(c-0)$ exist. So if c be any point of discontinuity, then that discontinuity is of first kind. In other words a monotone function can not have any discontinuity of second kind (for proof, see Apendix).

(ii) Polynomial function $a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n (a_i \in \mathbb{R}\ \forall\, i, a_0 \neq 0)$ is continuous on $\mathbb{R}$. Rational functions $\frac{p(x)}{q(x)}$ are continuous for all $x \in \mathbb{R}$ for which the functions can be defined.

$\sin x$ and $\cos x$ are continuous an $\mathbb{R}$. $\tan x$ & $\sec x$ are continuous for all $x \neq (2n+1)\frac{\pi}{2}$ and $\cot x$, $\operatorname{cosec} x$ are continuous for all $x \neq n\pi$ (n is integer in both cases)

(iii) $a^x, a > 0,$ is continuous for all $x \in \mathbb{R}$. $\log x, x > 0$ is continuous for all $x > 0$.

(iv) For even positive integer n, the function $g : x \to \sqrt[n]{x}$ is continuous for all $x \in [0, \infty)$ and for an odd positive integer n, g is continuous for all $x \in (-\infty, \infty)$.

(v) Limit of composite function :

Let $f : (a, b) \to \mathbb{R}$ be continuous at $c \in (a,b)$. Suppose that $g : I \to (a,b)$ where I is an open interval and $x_0 \in I$. If $\lim_{x \to x_0} g(x)$ exists and is equal to c. then $\lim_{x \to x_0} f(g(x)) = f(c)$.

Proof : Continuity of f at c implies that for each pre-assigned $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(y) - f(c)| < \varepsilon$ whenever $|y - c| < \delta,\ (y \in (a, b))$......(1)

As $\lim_{x \to x_0} g(x) = c$, so corresponding to above δ, we can find $\eta > 0$ such that $|g(x) - c| < \delta$ for $0 < |x - x_0| < \eta$ ....(2)

By (1) and (2) for $0<|x-x_0|<\eta$, we have $|f(g(x))-f(c)|<\varepsilon,\ 0<|x-x_0|<\eta$

Hence $\lim_{x\to x_0} f(g(x))=f(c)$ follows.

Corollary : Let I, J be open intervals, $g:I\to J$ be continuous at $x_0\in I$. If $f:J\to\mathbb{R}$ is continuous at $g(x_0)\in J$ then $f\circ g:I\to\mathbb{R}$ is continuous at x_0. In other words, the composition of two continuous functions is continuous.

Note : Continuity of f at c in (v) is needed.

Let $f,g:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(y)=\begin{cases}3, & y=1\\ 4, & y\neq 1\end{cases}\quad g(x)=1 \text{ for all } x.$$

Note that as $y\to 1, f(y)\to 4$ & $g(x)\to 1$ as $x\to 0$

For all x, $f(g(x))=f(1)=3$ & so it is not true that $f(g(x))\to 4$ as $x\to 0$

Illustration :

To evaluate $\lim_{x\to 1}\left(\frac{1+x}{2+x}\right)^{(1-\sqrt{x})/(1-x)}$

Let $f(x)=\frac{1+x}{2+x},\ g(x)=\frac{1-\sqrt{x}}{1-x}$

$\lim_{x\to 1} f(x)=\frac{2}{3}$ (f is continuous at $x=1$) & $\lim_{x\to 1} g(x)=\lim_{x\to 1}\frac{1}{1+\sqrt{x}}=\frac{1}{2}$

Hence $\lim_{x\to 1}\left[f(x)\right]^{g(x)}=\left(\frac{2}{3}\right)^{1/2}$

(Note that $\lim_{x\to 1}\left[f(x)\right]^{g(x)}=e^{\lim_{x\to 1} g(x)\ln f(x)}=e^{B\ln A}=A^B$ if

$\lim_{x\to 1} f(x)=A>0$ and $\lim_{x\to 1} g(x)=B$)

(vi) Piecewise Continuous function :

Let $f:[a,b]\to\mathbb{R}$ be such that it is continuous in $[a,b]$ except for a finite number of points, at each of which f has jump discontinuity. Then f is said to be piecewise continuous function in $[a,b]$

Illustration : Let $f:[0,3]\to\mathbb{R}$ be defined by $f(x)=[x]$

$$\text{Then } f(x)=\begin{cases}0, & 0\le x<1\\ 1, & 1\le x<2\\ 2, & 2\le x<3\\ 3, & x=3\end{cases}$$

Note that f has jump discontinuity at 1, 2, 3 only & is continuous in (0, 1), (1, 2) and (2, 3)

Example : Let $f(x)=[x],\ x\in\mathbb{R}^+$

Then f is not continuous at any point of $\mathbb{Z}$ but is continuous on $\mathbb{R}^+\setminus\mathbb{Z}$.

(i) Let $C\in\mathbb{Z}$.

Note that $C-\frac{1}{n}\to C$ as $n\to\infty$. $f\left(C-\frac{1}{n}\right)=C-1$ for all $n\in\mathbb{Z}$. But $f(C)=C$. So $\lim\limits_{n\to\infty} f\left(C-\frac{1}{n}\right)\neq f\left[\lim\limits_{n\to\infty}\left(C-\frac{1}{n}\right)\right]\Rightarrow f$ is not continuous at any point of $\mathbb{Z}$.

(ii) Let $C\in\mathbb{R}^+\setminus\mathbb{Z}$

We take $0<\varepsilon<\min\{C-[C],[C]+1-C\}$

Let $\lim\limits_{n\to\infty} x_n=C$. So corresponding to above ε, $\exists\, n_0\in\mathbb{Z}$ such that

$|x_n-c|<\varepsilon$ whenever $n\ge n_0$

Above choice of ε implies $[C] < x_n < [C]+1$ for all $n \geq n_0$

Then $f(x) = [x_n] = [C]$ for all $n \geq n_0$

Therefore $f(x_n) \to f(C)$ as $n \to \infty$. Hence the result follows.

Examples of piecewise continuous functions

(i) $f(x) = x - [x], x \in [0,4]$

(ii) $f(x) = \lim\limits_{n\to\infty} \dfrac{x^{2n}+5}{x^{2n}+1}, \ x \in [-2,2]$

(iii) $f(x) = \begin{cases} 2x+1, & 0 \leq x < 1 \\ 5, & x = 1 \\ 3x+2, & 1 < x \leq 2 \\ 7, & x = 2 \end{cases}$

2.4 Properties of functions continuous in a closed and bounded interval [a, b]

Theorem (1) : Let $f : [a,b] \to \mathbb{R}$ be continuous in the closed and bounded interval [a, b] & $f(a)f(b) < 0$. Then there exists at least one point $c \in (a,b)$ such that $f(c) = 0$.

[To prove this, we require the following result, known as Nested interval property :

If $\{[a_n, b_n]\}_n$ be a sequence of closed and bounded intervals such that each is contained in the preceeding. Then $\bigcap\limits_n [a_n, b_n] \neq \phi$

If more over $\lim\limits_{n\to\infty} (b_n - a_n) = 0$ then if $p \in \bigcap\limits_n [a_n, b_n]$, p is unique.]

Also $\lim_{n\to\infty} a_n = p = \lim_{n\to\infty} b_n$

Proof : We assume that $f(a) < 0, f(b) > 0$

For the sake of convenience, let $[a, b] = [a_1, b_1] \equiv I_1$

Let us bisect I_1 *at* $c_1 = \frac{a_1 + b_1}{2}$, If $f(c_1) = 0$ the result is proved

If $f(c_1) \neq 0$, either $f(c_1) > 0$ or $f(c_1) < 0$

If $f(c_1) > 0$ we take $[a_1, c_1]$ as I_2 so that $f(a_1) f(c_1) < 0$

& if $f(c_1) < 0$, we take $[c_1, b_1]$ as I_2. $I_2 \equiv [a_2, b_2]$

Let us bisect $[a_2, b_2]$ at $c_2 = \frac{a_2 + b_2}{2}$ If $f(c_2) = 0$ the result is proved. Otherwise, we assume that sub-interval as $[a_3, b_3] = I_3$ for which $f(a_3) f(b_3) < 0$ This process is continued indefinitely & we get a sequence $\{I_n\}_n$ of closed & bounded intervals $[a_n, b_n]$ for which

(i) $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$

(ii) $\lim_{n\to\infty} |I_n| = \lim_{n\to\infty} (b_n - a_n) = \lim_{n\to\infty} \frac{b-a}{2^{n-1}} = 0$

Also $f(a_n) f(b_n) < 0$ for all $n \in \mathbb{N}$

By Nested interval property, $\bigcap_n I_n = \{c\}$ Also $\lim_{n\to\infty} a_n = c = \lim_{n\to\infty} b_n$

By construction, $f(a_n) < 0$ and $f(b_n) > 0$ for all n

By continuity of f, $\lim_{n\to\infty} f(a_n) \leq 0$ & $\lim_{n\to\infty} f(b_n) \geq 0$

$\Rightarrow f\left(\lim_{n\to\infty} a_n\right) \leq 0$ & $f\left(\lim_{n\to\infty} b_n\right) \geq 0$

$\Rightarrow f(c) \leq 0$ & $f(c) \geq 0 \Rightarrow f(c) = 0$

Note : This theorem is due to B. P. J. N Bolzano (1781-1848)

Theorem (2) : Let $f:[a,b]\to\mathbb{R}$ be continuous in $[a,b]$ and $f(a)\neq f(b)$. If k be any real number such that $f(a)<k<f(b)$ then there exists $c\in(a,b)$ such that $f(c)=k$.

Proof : Let $\phi:[a,b]\to\mathbb{R}$ be defined by $\phi(x)=f(x)-k$

Continuity of f in $[a,b]\Rightarrow$ continuity of ϕ in $[a,b]$

$\phi(a)\,\phi(b)=\{f(a)-k\}\{f(b)-k\}<0$

Then by Bolzano's theorem, there exists $c\in(a,b)$ such that $\phi(c)=0$ i.e $f(c)=k$

Note : (i) This property is known as Intermediate value (I.V.) property of f in $[a, b]$

(ii) I. V. property does not hold in case of functions defined on a set.

Let $S=[0,1]\cup[2,3]$ & $f:S\to\mathbb{R}$ be defined by $f(x)=x$

f is continuous on S but f does not attain the value $\frac{3}{2}$ on S.

(iii) Continunity of f in $[a,b]\Rightarrow$ validity of I V property by f on $[a, b]$ but the converse is not true

Example : $f:[0,1]\to\mathbb{R}$ be defined by $f(x)=\begin{cases}0, & x=0\\ \frac{1}{2}-x, & 0<x<\frac{1}{2}\\ \frac{1}{2}, & x=\frac{1}{2}\\ \frac{3}{2}-x, & \frac{1}{2}<x<1\\ 1, & x=1\end{cases}$

f assumes every value between $f(0)$ & $f(1)$, f is not continuous in $[0,1]$ & so the validity of I. V property by a function in a closed & bounded interval does not characterise the continuity of the function. In this connection, we state the following two important results :

(1) Let $f : [a,b] \to \mathbb{R}$ obey the Intermediate value property in $[a,b]$ & let f be monotonic in $[a,b]$. Then f is continuous on $[a,b]$.

(2) Let f be stictly monotonic function in the interval $[a,b]$. If $f([a,b])$, the range set is an interval, then f is continous on $[a,b]$.

Theorem (3) : Let $f : [a,b] \to \mathbb{R}$ be continuous and assume each value between $f(a)$ and $f(b)$ just once. Then f is strictly monotonic in $[a,b]$.

Proof : Let $f(a) < f(b)$. We propose to show that f is strictly increasing function.

Let $a < x_1 < b$. As $f(x)$ assumes each value between $f(a)$ and $f(b)$ just once, so $f(x_1) = f(a)$ or, $f(x_1) = f(b)$ is not possible. ....(1)

If $f(x_1) < f(a)(< f(b))$, then by I. V. property $f(x)$ must assume the value $f(a)$ for some $x \in (x_1, b)$. As a result $f(x) = f(a)$, once at $x = a$ & for some $x \in (x_1, b)$. This contradicts the hypothesis that $f(x)$ assumes each value between $f(a)$ & $f(b)$ just once. So $f(x_1) < f(a)$ is not possible. ....(2)

By similar logic, $f(x_1) > f(b)$ is not possible. ....(3)

In that case, $f(x)$ assumes the value $f(b)$ at least twice — once at b & another in (a, x_1) by I. V. property.

By (1), (2) & (3), $f(a) < f(x_1) < f(b)$

This leads to the conclusion that if $a < x_1 < x_2 < b$ then

$f(a) < f(x_1) < f(x_2) < f(b)$

$\Rightarrow f$ is strictly monotonic increasing in $[a, b]$

If at the outset, we assume that $f(a) > f(b)$, then arguing in a similar way f is strictly monotonic decreasing in $[a, b]$.

Examples : $f:[0, 2] \to \mathbb{R}$ be defined by $f(x) = \lim_{n\to\infty} \dfrac{x^{2n+2} - \cos x}{1 + x^{2n}}$

Show that $f(0) f(2) < 0$ but $f(x)$ is never zero in $(0, 2)$. Explain why.

When 0 <x <1, $x^{2n} \to 0$ & when $1 < x < 2,\ x^{2n} \to \infty$

Here $f(0) = -1$. When $0 < x < 1,\ f(x) = -\cos x$

$f(1) = \frac{1}{2}[1 - \cos 1]$. When $1 < x \le 2,\ f(x) = \lim_{n\to\infty} \dfrac{x^2 - \frac{\cos x}{x^{2n}}}{1 + \frac{1}{x^{2n}}} = x^2$

$$\text{So } f(x) = \begin{cases} -1, & x = 0 \\ -\cos x, & 0 < x < 1 \\ \frac{1}{2}(1 - \cos 1), & x = 1 \\ x^2, & 1 < x \le 2 \end{cases}$$

So $f(0) f(2) = -4 < 0$, but $f(x)$ is never zero in $(0, 2)$. The reason is that f is not continuous in $[0, 2]$ & I. V. Property is not applicable.

(2) Let $f:\left[0, \frac{\pi}{2}\right] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \log(2 + x), & 0 \le x < 1 \\ \frac{1}{2}(\log 3 - \sin 1), & x = 1 \\ -\sin x, & 1 < x \le \pi/2 \end{cases}$$

Here $f(0)\ f\left(\frac{\pi}{2}\right)=(\log 2)(-1)<0$ but $f(x)$ is never zero in $\left(0, \pi/2\right)$. The reason is $f(x)$ is not continuous in $\left(0,\ \frac{\pi}{2}\right)$ & so I. V. property is not applicable here.

(3) Let $f:[0,1]\to\mathbb{R}$ be continuous function and assume only rational values in the entire interval. If $f(x)=5$ at $x=\frac{2}{3}$, show that $f(x)=5$ everywhere.

If possible, let there exist $c\in[0,1]$, $c\neq\frac{2}{3}$ and $f(c)\ =K\in\mathbb{R}$.

If $K\neq 5$, then by I. V. property of continuous function, $f(x)$ must assume every value between K & 5. Between K & 5, there are rational as well as irrational points also. But $f(x)$ assumes rational values only. So $f(x)\ =5$ throughout $[0,1]$.

(4) Let $f:[0,1]\to\mathbb{R}$ be continuous function and $f(0)=f(1)$. Show that there exists $y\in[0,1]$, such that $|x-y|=\frac{1}{2}$ and $f(x)=f(y)$.

Let us consider the function $g:\left[0,\frac{1}{2}\right]\to\mathbb{R}$ defined by $g(x)=f\left(x+\frac{1}{2}\right)-f(x)$

Continuity of f in $[0,1]\Rightarrow$ continuity of g in $\left[0,\frac{1}{2}\right]$.

$$g(0)\,g\left(\frac{1}{2}\right)=\left(f\left(\frac{1}{2}\right)-f(0)\right)\left(f(1)-f\left(\frac{1}{2}\right)\right)<0$$

By Bolzano's theorem on continuous function, there exists $c\in\left(0,\frac{1}{2}\right)$ such that

$g(c)=0 \Rightarrow f\left(c+\frac{1}{2}\right)=f(c)$ we get $x,\ y \in [0,1],\quad |x-y|=\frac{1}{2}$ for which

$f(x)=f(y)$.

(5) (Fixed point property) Let $f:[a,b]\to[a,b]$ be continuous function. Show that for some $\xi\in[a,\ b],\ f(\xi)=\xi$ holds.

If $f(a)=a$ or $f(b)=b$, the result is established.

We take $f(a)>a,\ f(b)<b.$ (as $f:[a,b]\to[a,b]$)

Let $g:[a,b]\to\mathbb{R}$ be defined by $g(x)=f(x)-x$

Continuity of f in $[a,b]\Rightarrow$ continuity of g in $[a,b]$.

$g(a)\,g(b)=\{f(a)-a\}\{f(b)-b\}<0$. So by Bolzano's theorem, there exists $\xi\in(a,\ b)$ such that $g(\xi)=0$ or $f(\xi)=\xi$.

Notes : (i) The condition of continuity of f can not be dropped

$f:[0,1]\to\mathbb{R}$ be defined by $f(x)=\begin{cases}1-x, & 0\le x\le\frac{1}{2}\\ \frac{1}{2}-\frac{x}{2}, & \frac{1}{2}<x\le 1\end{cases}$

(ii) The result may fail if the interval be not closed and bounded :

(a) $f:[0,1)\to\mathbb{R}$ be defined by $f(x)=\frac{1+x}{2}$

(b) $f:[1,\infty)\to\mathbb{R}$ be defined by $f(x)=x+\frac{1}{x}$

(iii) f must be defined on some interval $(\subset\mathbb{R})$

$f:S\to\mathbb{R}$ be defined by $f(x)=-x$ where $x\in S\ (\equiv[-2,-1]\cup[1,2])$

Also $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=x^2+1$

Exercise :

1. Show that $x.2^x = 1$ has a solution in $[0,1]$.

2. Let $f:[a,b] \to \mathbb{R}$ be continuous function & the equation $f(x)=0$ have finite number of roots in $[a,b]$ & arranging them in the ascending order, these are

$a < x_1 < x_2 < \ldots < x_{r-1} < x_r < \ldots x_{n-1} < b$

Prove that in each of (x_{r-1}, x_r) $f(x)$ must have the same sign.

3. If $f:[a,b] \to \mathbb{R}$ be a continuous function & $f(x)$ be always a rational number, then $f(x)$ is a constant function.

4. Examine for the continuity of $f: f(x) = \begin{cases} x^2 - 2x, & \text{when } x \text{ is rational} \\ 3x - 6, & \text{when } x \text{ is irrational} \end{cases}$

5. Does the equation $\sin x - x + 1 = 0$ have a root ?

6. Does the equation $f(x) = \dfrac{x^3}{4} - \sin \pi x + 3$ take on the value $2\dfrac{1}{3}$ within the interval $[-2,2]$?

7. Show that there exists $x \in \left(0, \dfrac{\pi}{2}\right)$ such that $x = \cos x$

Theroem (4) : Let $f:[a,b] \to \mathbb{R}$ be continuous in $[a,b]$. Then f is bounded in $[a,b]$ & attains its bounds in $[a,b]$.

Proof : If possible let f be not bounded in $[a,b]$. So corresponding to $n \in \mathbb{N}$, there exists $x_n \in [a,b]$ such that $|f(x_n)| \geq n$.

All such x_n's are in $[a,b]$. So we get a sequence $\{x_n\}_n$ in $[a,b]$. Hence $\{x_n\}_n$ is bounded in $[a,b]$.

By Bolzano-Weierstrass theorem on subsequence, there exists a convergent sub sequence $\{x_{r_n}\}_n$ (say) of $\{x_n\}_n$, which converges to $l(\in \mathbb{R})$. This $l \in [a, b]$ as $[a, b]$ is closed. Due to continuity of f, $\{f(x_{r_n})\}_n$ should converge to $f(x)$. Every convergent sequence is necessarily bounded. So $\{f(x_{r_n})\}_n$ is bounded. But by construction, $|f(x_{r_n})| \geq r_n$ & as $\{r_n\}_n$ is strictly increasing sequence of natural numbers, so $r_n \geq n$. Conequenctly, $|f(x_{r_n})| \geq n$

This contradicts $\{f(x_{r_n})\}_n$ is bounded.

This f is bounded on $[a, b]$

Let $M = \sup\limits_{[a,b]} f$, $m = \inf\limits_{[a,b]} f$

If possible, let there be no point x in $[a, b]$ at which $f(x) = M$. So $f(x) < M$ in $[a, b]$.

We construct $\phi : [a, b] \to \mathbb{R}$ defined by $\phi(x) = \dfrac{1}{M - f(x)}$ for all $x \in [a, b]$.

Continuity of f in $[a, b] \Rightarrow$ Continuity of ϕ in $[a, b]$. So ϕ is bounded in $[a, b]$. Let $G > 0$ be any number, as large as we please.

As $M = \sup\limits_{[a,b]} f$, there exists at least one point $\xi \in [a, b]$ such that

$$f(\xi) > M - \frac{1}{G}$$

$\Rightarrow \dfrac{1}{M - f(\xi)} > G \Rightarrow \phi(\xi) > G$. This contradicts the fact that ϕ is bounded in $[a, b]$

So there exists a point in $[a, b]$ at which $f(x) = M$.

Similarly, it can be shown that there exists a point in $[a, b]$ at which $f(x) = m$ holds.

Corollaries : (i) If $f:[a, b] \to \mathbb{R}$ be a non-constant continuous function, then $f(x)$ assumes every value between its infimum & supremum.

By above theorem, there are points $\xi, \eta \in [a, b]$ such that $f(\xi) = M$, $f(\eta) = m$. By I. V. property of continuous function, applied to f in $[\xi, \eta]$ (or $[\eta, \xi]$) the result follows.

(ii) Let $I (\subset \mathbb{R})$ be a closed and bounded interval & let $f: I \to \mathbb{R}$ be non constant continuous function in I.

Then the set $f(I) = \{f(x) : x \in I\}$ is a closed & bounded interval.

If $M = \sup_{[a, b]} f$, $m = \inf_{[a, b]} f$, then $m \le f(x) \le M$ for all $x \in I$

$\Rightarrow f(I) \subseteq [m, M]$ ...1

Let k be any element of $[m, M]$. Then by Corollany 1, there exists $c \in I$ such that $f(c) = k \in f(I)$

So $[m, M] \subseteq f(I)$ ... (2)

By (1) and (2), $f(I) = [m, M]$

Note : The result fails if the condition of continuity be dropped.

$f : I \equiv [-1, 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} \frac{|x|}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

$f(I)$ is not an interval.

2. The continuous image of an open interval may not be open.

Let $f : (-1, 1) \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x^2 + 1}$

Here $f(I)=(\frac{1}{2},1]$ which is not open interval

3. The continuous image of an unbounded closed interval may not be closed.

Let $f: I\equiv[0,\infty)\to\mathbb{R}$ be defined by $f(x)=\dfrac{1}{x^2+1}$

Here $f(I)=(0,1]$ which is not closed.

Example (1) : Let $f:[a,b]\to\mathbb{R}$ be continuous in $[a,b]$

Let $x_1\, x_2,......x_n\in[a,b]$. Show that there exists a point ξ in $[a,b]$ such that

$$f(\xi)=\frac{1}{n}\sum_{i=1}^{n}f(x_i)$$

As f is continuous in $[a,b]$, there are points $\alpha,\beta\in[a,b]$ such that

$f(\alpha)\le f(x)\le f(\beta)$ for all $x\in[a,b]$

$$\Rightarrow nf(\alpha)\le\sum_{i=1}^{n}f(x_i)\le nf(\beta)$$

$$\Rightarrow f(\alpha)\le\frac{1}{n}\sum_{i=1}^{n}f(x_i)\le f(\beta)$$

By I.V. property of continuous functions, there exists $\xi\in[a,\, b]$ such that

$$f(\xi)=\frac{1}{n}\sum_{i=1}^{n}f(x_i)$$

(2) Let $f,g:\mathbb{R}\to\mathbb{R}$ are continuous on $\mathbb{R}$. Show that

$A=\{x\in\mathbb{R}\,|f(x)>g(x)\}$, $B=\{x\in\mathbb{R}|f(x)<g(x)\}$, $C=\{x\in\mathbb{R}|f(x)\ne g(x)\}$

are open sets in $\mathbb{R}$ whereas $D=\{x\in\mathbb{R}\,|f(x)\equiv g(x)\}$ is a closed set in $\mathbb{R}$.

Let $\phi(x)=f(x)-g(x)$, $x\in\mathbb{R}$. As f, g are continuous, so $\phi(x)$ is continuous in $\mathbb{R}$.

(i) $A=\{x\in\mathbb{R}|\phi(x)>0\}$

Case I : If $\phi(x) \le 0$ in $\mathbb{R}$. Then $A = \phi$ & So A is open set in $\mathbb{R}$.

Case II : If $\phi(x) > 0$ in $\mathbb{R}$. So $A \equiv \mathbb{R}$ & $\mathbb{R}$ being open set, A is open set in $\mathbb{R}$.

Case III : Let $A \subset \mathbb{R}$.

Let $p \in A$, So $\phi(p) > 0$ & by neighbourhood property of continuous function, there exists $\delta > 0$ such that $x \in \mathbb{N}(p, \delta) \Rightarrow \phi(x) > 0$

Thus $\mathbb{N}(p, \delta) \subset A$ & so p is interior point of A. This is true for all $p \in A$. Consequently A is open set in $\mathbb{R}$.

Arguing in a similar way, B is open set in $\mathbb{R}$.

Set $C = A \cup B$ so C is union of two open sets in $\mathbb{R}$ & so C is open set in $\mathbb{R}$. D is the complement of open set C & hence D is closed.

(3) Let $I(\subset \mathbb{R})$ be a given open interval. Let $f : I \to \mathbb{R}$ be continuous on I. Let α be an arbitrary real constant.

Then $I[f < \alpha] \equiv \{x \in I : f(x) < \alpha\}$ and $J[f > \alpha] = \{x \in I : f(x) > \alpha\}$ are open sets.

If $f(x) = \alpha$ for all x, I and J are void sets & so are open sets in $\mathbb{R}$.

Next let $I[f(x) < \alpha] \ne \phi$

So there exists $p \in I$ i.e. $f(p) < \alpha$. Let $0 < \varepsilon < \frac{1}{2}[\alpha - f(p)]$.

Continuity of f at $p \Rightarrow$ corresponding to above chosen ε, there exists $\delta > 0$ such that $|f(x) - f(p)| < \varepsilon$ whenever $x \in N(p, \delta) \cap I$ ...(1)

By hypothesis, I is open set & p is interior point of I. By definition of interior point, there exists $r, 0 < r < \delta$, such that $N(p, r) \subset I$ ...(2)

By (1) & (2), $f(x) < f(p) + \varepsilon < f(p) + \frac{1}{2}(\alpha - f(p))$

$\Rightarrow f(x) < \alpha$ where $x \in N(p, r) \Rightarrow N(p, r) \subset I[f < \alpha]$

$\Rightarrow I[f < \alpha]$ is an open set in $\mathbb{R}$.

Following similar argument, $J[f > \alpha]$ is also open set in $\mathbb{R}$.

(iv) Let $f, g : [0,1] \to [0, \infty)$ be continuous functions satisfying

$$\sup_{[0,1]} f(x) = \sup_{[0,1]} g(x)$$

Show that there exists $c \in [0,1]$ such that $f(c) = g(c)$

Continuity of f, in $[0,1] \Rightarrow$ boundedness & their attainment of bounds in [0,1].

Let $M = \sup_{[0,1]} f(x) = \sup_{[0,1]} g(x)$

If both f & g attain M at the same point, the result is established.

Otherwise : Let $f(\xi) = M$ and $g(\eta) = M$ for some $\xi, \eta \in [0,1]$,

So $g(\xi) < M$, $f(\eta) < M$.

We construct $h : [0,1] \to \mathbb{R}$ by $h(x) = f(x) - g(x)$. Then h is continuous in [0, 1] & by above $h(\xi) = f(\xi) - g(\xi) = M - g(\xi) > 0$ and

$h(\eta) = f(\eta) - g(\eta) = f(\eta) - M < 0$. So $h(\xi)h(\eta) < 0$.

$\Rightarrow$ By Bolzano's theorem, there exists $c \in (\xi, \eta) \subset (0,1)$ such that $h(c) = 0$

or in other words, $f(c) = g(c)$.

Continuity of Inverse function :

Theorem : Let $f : [a, b] \to \mathbb{R}$ be strictly monotonic and continuous on the closed and bounded interval $[a, b]$. Then there exists an inverse function $g : f[a, b] \to \mathbb{R}$ such that (i) g is strictly monotonic in $f[a, b]$ and (ii) g is continuous in $f[a, b]$

Proof : Let f be strictly increasing in $[a, b]$ ..... (1)

Continuity of f in $[a,b] \Rightarrow$ boundedness of f in $[a,b]$ & attainment of bounds in [a, b]. So $\sup\limits_{[a,b]} f = f(b)$, $\inf\limits_{[a,b]} f = f(a)$.

Therefore, here $f([a,b]) = [f(a), f(b)]$... (1)

As f is strictly increasing, so for any distinct pair of points $x_1, x_2 \in [a,b]$, $f(x_1) \neq f(x_2) <\Rightarrow x_1 \neq x_2$. So f is injective. ... (2)

Consequently by (1) & (2) f is bijective. So $f^{-1} = g$ exists where $g : f([a,b]) \to [a,b]$. where $f(x) = y \Rightarrow x = g(y), x \in [a,b], y \in f[a,b]$

Let $y_1, y_2 \in f[(a,b)]$. So there are $x_1, x_2 \in [a,b]$ such that

$y_1 = f(x_1), y_2 = f(x_2)$

f being strictly increasing in $[a,b], y_1 < y_2 \Rightarrow x_1 < x_2$

As a result, $y_1 < y_2 \Rightarrow g(y_1) < g(y_2) \Rightarrow g$ is strictly increasing in $f([a,b])$.

Let y_0 be any point between $f(a)$ and $f(b)$ & x_0 be the corresponding value of x.

Let $\varepsilon > 0$ be arbitrary number such that $x_0 - \varepsilon$, $x_0 + \varepsilon$ are in $[a,b]$. Let $g(y_0 - \eta_1) = x_0 - \varepsilon$ and $g(y_0 + \eta_2) = x_0 + \varepsilon$ such that $\eta_1, \eta_2 > 0$ exist by above.

Let η be such that $0 < \eta < \min\{\eta_1, \eta_2\}$. Then

$|x - x_0| < \varepsilon$ whenever $|y - y_0| < \eta, \eta$ depends on ε.

So $g(y)$ is continuous at y_0. and this is true for all $y_0 \in [f(a), f(b)]$

Hence the result follows :

Note (i) Continuity of Inverse function is preserved only when the domain is closed and bounded.

Let $A = [0,1) \cup [2,3]$ and $f : A \to \mathbb{R}$ be defined by

$$f(x)=\begin{cases} x, & x\in[0,1) \\ x-1, & x\in[2,3] \end{cases}$$

$$f^{-1}(x)=\begin{cases} x, & x\in[0,1) \\ x+1, & x\in[1,2] \end{cases} \Rightarrow f^{-1} \text{ is discontinuous at } x = 1.$$

Theorem : If $f:[a,b]\to\mathbb{R}$ be continuous, injective function, then f is strictly monotone function.

If possible, let f be not strictly monotone function in $[a,b]$ though f is continuous & injective in $[a,b]$. So we say that there are three points $p,q,r\in[a,b]$ where $p<q<r$ nonetheless $f(q)$ does not lie between $f(p)$ and $f(r)$. Consequently, either $f(r)$ lies between $f(p)$ and $f(q)$ or $f(p)$ lies between $f(q)$ and $f(r)$. For definiteness, let $f(p)$ be between $f(q)$ and $f(r)$.

By hypothesis, f is continuous in $[q,r]\subset[a,b]$. By I. V. property, there exists $s\in(q,r)$ such that $f(s)=f(p)$.

So $p<s$ but $f(p)=f(s)$. This contradicts the injectivity of f.

Similarly if we assume that $f(r)$ lies between $f(p)$ and $f(q)$, we would arrive at same type of contradiction. So f is strictly monotone.

Corollary : A continuous function $f:[a,b]\to\mathbb{R}$ is injective if and only if f is strictly monotone in $[a,b]$.

Example : Assume that $f:\mathbb{R}\to\mathbb{R}$ satisfies $f(f(x))=f^2(x)=-x$ for all $x\in\mathbb{R}$.

Then f can not be continuous.

First we propose to show that f is injective.

$f(x_1)=f(x_2)\Rightarrow f^2(x_1)=f^2(x_2)\Rightarrow -x_1=-x_2\Rightarrow x_1=x_2$

If f be continuous then it would be either strictly increasing or strictly decreasing. In both cases, f^2 would be increasing.

For if $p<q$, then $f(p)<f(q)$ (in case f is increasing) & $f(p)>f(q)$ (in case f is decreasing). In the first case, $f(f(p))<f(f(q))$ & in the second case $f(f(p))<f(f(q))$. So in any case, $f^2(p)<f^2(q)$

$\Rightarrow -p<-q$ absurd as $p<q$

So f can not be continuous.

Exercise :

1. Let $f:[0,1]\to\mathbb{R}$ be defined by $f(x)=\begin{cases}2x-1, \text{ if } x\in(0,1)\\ 0, \quad \text{if } x=0 \text{ or } 1\end{cases}$

Choose the correct answer :

(a) f is unbounded function (b) f is bounded function and attains its bounds there in (c) f is bounded function but does not attain its bounds.

2. Let $f(x)=\begin{cases}2^x+1, & \text{for } -1\le x<0\\ 2^x, & \text{for } x=0\\ 2^x-1, & \text{for } 0<x\le 1\end{cases}$

Choose the correct answer :

(a) f is bounded in $[-1,1]$

(b) f is unbounded in $[-1,1]$

(c) f is continuous in $[-1,1]$

(d) f has jump discontinuity in $[-1,1]$

2.5 Uniform Continuity

Recall our $\varepsilon-\delta$ definition of continuity of function. The following example will illustrate that the δ mentioned in the definition depends not only on ε, but on the point also.

$f(x)=x^2$ is continuous on $\mathbb{R}$. Let us consider the continuity of x^2 at $x=0$.

Let $\varepsilon=\frac{1}{9}$. Then $|f(x)-f(0)|<\frac{1}{9}\Rightarrow|x|<\frac{1}{3}$. So we get $\delta=\frac{1}{3}$ such that

$|f(x)-f(0)|<\frac{1}{9}$ wherever $|x-0|<\delta$. Our point is that $\delta=\frac{1}{3}$ is permissible here.

Let us examine whether this δ serves for all points of $\mathbb{R}$.

$f(x)$ is continuous at $x=1$. If the above δ serves for $x=1$ also, we would have $|f(x)-f(1)|<\frac{1}{9}$ whenever $|x-1|<\delta\left(=\frac{1}{3}\right)$.

Note that $x=1.3$ satisfy $|x-1|<\frac{1}{3}$. But then

$$|f(x)-f(1)|=.69 \nless \delta\left(=\frac{1}{3}\right)$$

So the δ, obtained in case of $x=0$, does not serve the purpose for $x=1$

Let us consider another example $f(x)=\frac{1}{x}$ in $(0,1)$

If possible, let there exist $\delta>0$ such that $|f(x)-f(y)|<1$ wherever $|x-y|<\delta,\ x,y\in(0,1)$

Let $x=\frac{\delta}{1+\delta}, y=\frac{\delta}{2(1+\delta)}$ $(\text{both}\in(0,1))$.

Note that for these, $|x-y|=\frac{\delta}{2(1+\delta)}<\delta$.

But $|f(x)-f(y)|=\frac{1+\delta}{\delta}>1$

So the above δ is not applicable here.

Our observation is that the δ, appeared in the $\varepsilon-\delta$ definition for continuity, depends both on ε and the point itself. At this stage, our purpose is to investigate whether there exists $\delta>0$ which depends only on ε so that the δ can serve for all points of D_f.

Let $A=\{\delta(p,\varepsilon): p\in D_f\}$ where each $\delta>0$. This set A is non-void bounded below subset of $\mathbb{R}$ and has inf δ_0(say). Then $\delta_0\geq 0$.

If $\delta_0>0$, then for any $p\in D_f$, $|x-p|<\delta_0 \Rightarrow |f(x)-f(p)|<\varepsilon$. So this δ_0 serves for all points of D_f. As the δ_0 serves for all points of D_f, then the continuity is known as **uniform continuity** & f is said to be uniformly continuous on D_f.

(Note : Uniform Continuity is of global character)

Definition : A function $f:D\to\mathbb{R}(D\subset\mathbb{R})$ is said to be uniformly continuous on *D*, if given $\varepsilon>0$, there exists $\delta>0$. depending on ε only, such that for any pair of points *x*, *y* of D satisfying $|x-y|<\delta$, we have $|f(x)-f(y)|<\varepsilon$

Example I : if $f:D\to\mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on *D*.

As $f:D\to\mathbb{R}$ is a Lipschitz function, there exists a constant $\lambda>0$ such that

$|f(x)-f(u)|\leq\lambda|x-u|$ for all $x,u\in D$

Let $\varepsilon>0$ be any number. Taking $\delta=\dfrac{\varepsilon}{\lambda}$, we get

$|f(x)-f(u)|<\varepsilon$ for all $x,y\in D$ satisfying $|x-u|<\delta$

$\Rightarrow f$ is uniformly continuous on *D*.

Uniform Continuity in closed and bounded interval $[a,b]$.

Theorem : Let $f:[a,\ b]\to\mathbb{R}$ be continuous in $[a,\ b]$. Let $\varepsilon>0$ be any number. Then the interval $[a,b]$ can be divided into finite number of sub-intervals in such a way that

$|f(x_2)-f(x_1)|<\varepsilon$ whenever x_1 & x_2 are any two points in the same sub-interval.

Proof : If possible, let the theorem be false in $[a, b] \equiv [a_1, b_1]$.

We bisect $[a_1, b_1]$ of $c_1 = \dfrac{a_1 + b_1}{2}$. Then the theorem is false in at least one of $[a_1, c_1]$ and $[c_1, b_1]$. We designate that sub-interval as $[a_2, b_2]$ in which the theorem is false. Again we bisect $[a_2, b_2]$ at $c_2 = \dfrac{a_2 + b_2}{2}$ & let $[a_3, b_3]$ be the sub-interval in which the theorem is false.

Proceeding in this way, we obtain a sequence of nested intervals $\{[a_n, b_n]\}_n$ such that (i) each is contained in the preceeding (ii) $\lim\limits_{n\to\infty}(b_n - a_n) = \lim\limits_{n\to\infty}\dfrac{b-a}{2^{n-1}} = 0$

Also the theorem is false in each $[a_n, b_n]$.

By Nested interval theorem, $\exists \xi \in [a_n, b_n]$ for all n, ξ is unique and $\lim\limits_{n\to\infty} a_n = \xi = \lim\limits_{n\to\infty} b_n$

I. $a < \xi < b$

By hypothesis, f is continuous at ξ. So given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(\xi)| < \dfrac{\varepsilon}{2}$ wherever $|x - \xi| \le \delta$.

As $\lim\limits_{n\to\infty}(b_n - a_n) = 0$ and $\xi \in [a_n, b_n]$ for all n, so for sufficiently large n, say for $n \ge n_0 (\in \mathbb{N}), [a_n, b_n]$ lies wholly in $[\xi - \delta, \xi + \delta]$

Let x_1, x_2 be any two distinct points in $[a_n, b_n]$ for $n = n_0$

so $|f(x_1) - f(\xi)| < \dfrac{\varepsilon}{2}$, $|f(x_2) - f(\xi)| < \dfrac{\varepsilon}{2} \Rightarrow |f(x_2) - f(x_1)| < \varepsilon$

So the theorem is true in $\left[a_{n_0}, b_{n_0}\right]$. Thus we arrive at a contradiction.

II. Let $\xi = a$

Arguing as before and noting that for sufficiently large values of n. $[a_n, b_n] = [a, b_n] \subseteq [a, a+\delta]$, we will arrive at a similar type of contradiction.

III. Let $\xi = b$

Here for sufficiently large values of n,

$[a_n, b_n] = [a_n, b] \subseteq [b-\delta, b]$ & arguing as before, we will arrive at a similar type of contradiction.

Hence the theorem follows.

Corollaries (I) Let δ be the least of the lengths of the sub-intervals mentioned above.

Let us consider two points x_1, x_2 of $[a, b]$ such that $|x_1 - x_2| < \delta$. Then two cases may arise :

(i) x_1 and x_2 belong to the same sub interval

(ii) x_1 and x_2 belong to two consecutive sub-intervals.

(i) In this case, by the theorem, $|f(x_2) - f(x_1)| < \varepsilon$ holds.

(ii) Let c be the point which separates the two sub intervals.

Then x_1, c are in one sub interval & c, x_2 are in another same subinterval.

So by the theorem, $|f(x_1) - f(c)| < \varepsilon/2$ and $|f(c) - f(x_2)| < \varepsilon/2$

As a result, $|f(x_2) - f(x_1)| \le |f(x_2) - f(c)| + |f(x_1) - f(c)| < \varepsilon$ holds.

So given $\varepsilon > 0$, there exists $\delta > 0$ such that if

$|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \varepsilon$ holds.

(II) Let η_r, denote any sub-interval of $[a, b]$ such that the length of η_r is less then δ, where $\delta > 0$ is as above. If x_1, x_2 be any two points of η_r then $|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$

let $M_r = \sup_{\eta_r} f,\ m_r = \inf_{\eta_r} f$. As f is continuous in η_r, these bounds are attained in η_r. Let x' and x'' be the two points in η_r where $M_r = f(x'), m_r = f(x'')$

Then by above, $M_r - m_r < \varepsilon$.

So if f is continuous in $[a, b]$ & ε be any positive number, there exists $\delta > 0$ such that the oscillation of f in every sub-interval of length less than δ, is less than ε.

Theorem : If f is continuous in the closed and bounded interval $[a, b]$, then f is uniformly continuous in $[a, b]$.

Proof : If possible, let f be not uniformly continuous in $[a, b]$. Hence there exists $\varepsilon_0 > 0$ for which there is no $\delta > 0$ with the property that

$|f(x_2) - f(x_1)| < \varepsilon_0$ for all pair of points x_1, x_2 of $[a, b]$ satisfying $|x_1 - x_2| < \delta$. In other words, for all each positive integer n, there is a pair x'_n, x''_n of $[a, b]$ such that $|x'_n - x''_n| < \frac{1}{n}$ nonetheless $|f(x'_n) - f(x''_n)| \geq \varepsilon_0$ .... (1)

As $x'_n \in [a, b]$ for all n, $\{x'_n - x''_n\}_n$ is a bounded sequence in $\mathbb{R}$. By Bolzano-Weierstrass theorem on subsequence, there is a subsequence $\{x'_{k_n}\}_n$ of $\{x'_n\}_n$ which converge to x_0 and $x_0 \in [a, b]$ as $[a, b]$ is closed.

Since $\left|x'_{k_n} - x''_{k_n}\right| < \frac{1}{n}$, we see that $x''_{k_n} \to x_0$ as $n \to \infty$

$\left(\left\{x''_{k_n}\right\} \text{is subsequence of } \{x''_n\}_n\right)$

Due to continuity of f, $f\left(x'_{k_n}\right) \to f(x_0), f\left(x''_{k_n}\right) \to f(x_0)$.

So coresponding to above ε_0, there are natural numbers m_1, m_2 such that

$\left|f\left(x'_{k_n}\right)-f\left(x_0\right)\right|<\frac{\varepsilon}{2}$ for all $n \geq m_1$ & $\left|f\left(x''_{k_n}\right)-f\left(x_0\right)\right|<\varepsilon/2$ for $n \geq m_2$.

Hence for all $n \geq m = \max\{m_1, m_2\}$, both hold & we get $\left|f\left(x'_{k_n}\right)-f\left(x''_{k_n}\right)\right|<\varepsilon_0$ for all $n \geq m$.

This last inequality is in contradiction to (1). Hence f is uniformly continuous in $[a,b]$

Uniform Continuity in open interval (*a*, *b*)

Theorem : Let f be continuous in (a,b), then f is uniformly continuous in (a,b) if and only if $\lim\limits_{x\to a+0} f(x)$ and $\lim\limits_{x\to b-0} f(x)$ both exist finitely.

Proof : Let f be continuous in the bounded open interval (a, b) and $\lim\limits_{x\to a+} f(x)$ and $\lim\limits_{x\to b-} f(x)$ both exist finitely.

We construct $g:[a,b]\to\mathbb{R}$ as follows :

$$g(x)=f(x) \text{ for all } x\in(a,b)$$

$$g(a)=\lim_{x\to a+} f(x) \text{ and } g(b)=\lim_{x\to b-} f(x)$$

Then $\lim\limits_{x\to a+} g(x)=\lim\limits_{x\to a+} f(x)=g(a)$ and $\lim\limits_{x\to b-0} g(x)=\lim\limits_{x\to b-} f(x)=g(b)$

Along with this, considering the continuity of f in (a,b), we say that g is continuous in $[a,b]$ & so g is uniformly continuous in $[a,b]$ & (a,b). But g and f are identical in (a,b). So f is uniformly continuous in (a,b).

Converse : Let f be uniformly continuous in open interval (a,b).

We propose to show that both $\lim\limits_{x\to a+} f(x)$ & $\lim\limits_{x\to b-} f(x)$ exist finitely.

If possible, suppose that $\lim\limits_{x\to a+} f(x)$ does not exist. Then there is a sequence

$\{x_n\}_n$ in (a,b) with $x_n \to a$ such that the sequence $\{f(x_n)\}_n$ does not converge & hence is not a cauchy sequence in $\mathbb{R}$. Then there exists some $\varepsilon_0 (> 0)$ with the property that there is no natural number n_0 for which

$$i,j \geq n_0 \Rightarrow \left|f(x_i) - f(x_j)\right| < \varepsilon_0 .$$

Consequently we can find arbitrary large $i, j \in \mathbb{N}$ for which $\left|f(x_i) - f(x_j)\right| \geq \varepsilon_0$. Now since the sequence $\{x_n\}_n$ is a cauchy sequence in $\mathbb{R}$, we have $\lim\limits_{i,j\to\infty} \left|x_i - x_j\right| = 0$. Clearly for this ε_0, we can find a pair of points $x_i, x_j \in (a,b)$ which are arbitrarily close and for which $\left|f(x_i) - f(x_j)\right| \geq \varepsilon_0$. This implies that f is not uniformly continuous in (a,b).

A similar argument can be in the case when $\lim\limits_{x\to a+} f(x)$ exists but $\lim\limits_{x\to b-} f(x)$ fails to exist.

Illustration : Let $f(x) = \dfrac{1}{x}$ in $(0,1)$. $\lim\limits_{x\to 0+} f(x)$ does not exist finitely & so f is not uniformly continuous in $(0,1)$.

An imporant non-uniform continuous criteria

A function $f : D \to \mathbb{R} (D \subset \mathbb{R})$ is not uniformly continuous on D if and only if there exist sequences $\{x_n\}_n$ and $\{t_n\}_n$ in D such that

(i) $|x_n - t_n| \to 0$ (ii) $|f(x_n) - f(t_n)| \nrightarrow 0$

Examples : (a) $\dfrac{1}{x}$ is not uniformly continuous in $(0,1)$

choose the sequences $x_n = \dfrac{1}{n}$ and $t_n = \dfrac{1}{2n}, n \in \mathbb{N}$

(b) x^2 is not uniformly continuous on $\mathbb{R}$

choose the sequences $x_n = n,\ t_n = n + \frac{1}{n},\ n \in \mathbb{N}$

(c) $\sin\frac{1}{x}$ is not uniformly continuous in $(0, \infty)$

choose the sequences $x_n = \frac{1}{2n\pi}, t_n = \frac{2}{(4n+1)\pi}, n \in \mathbb{N}$

(d) $\sin x^2$ is not uniformly continuous on $\mathbb{R}$.

choose the sequences $x_n = \sqrt{\frac{\pi}{2}(n+1)}$ & $t_n = \sqrt{\frac{\pi}{2}n}$, $n \in \mathbb{N}$

(e) $e^{1/x}$ is not uniformly continuous on $(0, 1)$.

Choose $x_n = \frac{1}{\ln n},\ t_n = \frac{1}{\ln(n+1)},\ n \in \mathbb{N} - \{1\}$

(f) $x \sin x$ is not uniformly continuous on $(0,\infty)$

choose $x_n = 2n\pi,\ t_n = 2n\pi + \frac{1}{n},\ n \in \mathbb{N}$

Examples :

(1) If $f, g : D \to \mathbb{R}(D \subset \mathbb{R})$ be both uniformly continuous on D & D be bounded, then fg is uniformly continuous on D.

To solve this we will use the following result (which is being stated here without proof).

If $f : D \to \mathbb{R}(D \subset \mathbb{R})$ be uniformly continuous on a bounded set D, then f is bounded on D.

f, g are bounded on D. So there exists $\lambda \in \mathbb{R}^+$ such that $|f(x)| < \lambda, |g(x)| < \lambda$ for all $x \in D$.

Let $x, y \in D$ then

$$|f(x)g(x)-f(y)g(y)| \le |f(x)||g(x)-g(y)|+|g(y)||f(x)-f(y)| \quad ..... (1)$$

Let $\varepsilon > 0$ any number. As *f, g* are uniformly continuous on *D*, corresponding to above ε, there exists $\delta_1 > 0, \delta_2 > 0$, both depend on ε only, such that for any pair of points *x, y* of *D* satisfying $|x-y| < \delta_1$, we have $|f(x)-f(y)| < \frac{\varepsilon}{2\lambda}$ and

$$|x-y| < \delta_2 \Rightarrow |g(x)-g(y)| < \frac{\varepsilon}{2\lambda} \quad ....(2)$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Recalling (1) & (2)

$|f(x)g(x)-f(y)g(y)| < \lambda.\frac{\varepsilon}{2\lambda}+\lambda.\frac{\varepsilon}{2\lambda}$ for any pair of points *x, y* of *D* satisfying $|x-y| < \delta$

$\Rightarrow fg$ is uniformly continuous on D.

Note the result fails if *D* be not bounded. This is evident from the example x^2 on $\mathbb{R}$.

(2) Every uniformly continuous function maps a cauchy sequence onto a cauchy sequence.

Let $\{x_n\}_n$ be a cauchy sequence in $\mathbb{R}$.

Let $\varepsilon > 0$ be any number. Since f is uniformly continuous on D, corresponding to above ε, there exists $\delta > 0$ (δ depends only on ε) such that for any pair of points *x, y* of *D* that satisfy $|x-y| < \delta$, we have $|f(x)-f(y)| < \varepsilon$ ......(1)

Since $\{x_n\}_n$ is a cauchy sequence in $\mathbb{R}$, corresponding to above δ, there exists $m \in \mathbb{N}$ such that $|x_{n+p}-x_n| < \delta$ for all $n \ge m, p \in \mathbb{N}$ ....(2)

By (1) and (2), $|f(x_{n+p})-f(x_n)| < \varepsilon$ for all $n \ge m, p \in \mathbb{N}$

$\Rightarrow \{f(x_n)_n\}$ is a cauchy sequence.

Note : The result fails if *f* be only continuous on *D*.

Consider $f(x)=\frac{1}{x}$ in $(0,1)$ and $x_n=\frac{1}{n}(n\in\mathbb{N})$.

Here $f(x_n)=n$ and $\{f(x_n)\}_n$ is not cauchy sequence.

2.6 Summary

In this unit, we have defined the terms continuity and discontinuity and given various examples. We have studied various types of discontinuities and their properties. We have explained the most important properties of functions continuous in a closed and bounded interval [a, b], such as, Intermediate value property, Fixed point property. We have also shown the relation between continuity and monotonicity. We have further study the maximum-minimum property. We have introduced the notion of uniform continuity and shown that in a closed and bounded interval [a, b] this concept is same with the concept of continuity. We also studied the uniform continuity on an open interval (a, b), and give an important non-uniform continuity criteria. We have also shown that every uniformly continuous function maps cauchy sequence into a cauchy sequence.

2.7 Exercise

1. Prove or disprove : If $f:S\to\mathbb{R}$, $g:T\to\mathbb{R}\ (S,T\subset\mathbb{R})$ are uniformly continuous and $f(S)\subset T$, then the composite function $g\circ f:S\to\mathbb{R}$ is uniformly continuous on S.

2. Show that $e^x\cos\frac{1}{x}$ is not uniformly continuous on $(0,1)$.

(Hints : You can consider the sequences $\left\{\frac{1}{2n\pi}\right\}_n$ & $\left\{\frac{1}{(2n+1)\pi}\right\}_n$)

3. Let $f(x)=\sqrt{x},\ x\in[0,2]$

Choose the correct answer(s) :

(i) f is Lipschitz function in $[0,2]$

(ii) f is not Lipschitz function in $[0,2]$

(iii) f is uniformly continuous in $[0, 2]$

(iv) f is not uniformly continuous in $[0, 2]$

4. Correct or justify : $x \sin^2 x$ is uniformly continuous on $\mathbb{R}$.

5. Let $f:[0,1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x \cos \dfrac{\pi}{2x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Examine whether f is uniformly continuous on $[0, 1]$.

6. Let $f:[a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and let the equation $f(x) = 0$ have finite number of roots in $[a,b]$. Arrange them in the ascending order.

$a < x_1 < x_2 < \ldots < x_{r-1} < x_r < \ldots. < x_n < b$

Prove that in each of the intervals $(a_1, x_1), (x_1, x_2), (x_{r-1}, x_r) (x_n, b)$ the function $f(x)$ retains the same sign.

Unit-3 □ Differentiation of Functions

Structure

3.0 Objectives

This unit gives

- The concept of differentiation of a function
- Algebric operation of differentiable function
- Rolle's theorem and some application
- Expansion of a differentiable function in series form

3.1 Introduction

The problem of finding tangent lines and the seemingly unrelated problem of finding maximum or minimum values were first seen to have a connection by Fermat in the 1630s. And the relation between tangent lines to curve and the velocity of a moving particle was discovered in the late 1660s by Isaac Newton. Newton's theory of 'fluxions' which was based on an intuitive idea of limit. But the vital observation, made by Newton and, independently, by Gottfried Leibniz in the 1680s, was that areas under curves could be calculated by reversing the differentiation process. In this chapter we will develop the theory of differentiation.

3.2 Differentiation of Functions

The derivative : Let $f : D \to \mathbb{R}(D \subset \mathbb{R})$ be a given function, c be a point of D as well as an accumulation point of D. So the function $x \to \frac{f(x)-f(c)}{x-c}$ is defined on $D-\{c\}$.

If $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ exists finitely and be $=L(\in\mathbb{R})$ then we say that f is derivable at c, $f'(c)$ exists and $= L$

If $f : [a,b]\to\mathbb{R}$ then $f'(a)$ is in fact $Rf'(a)=\lim_{x\to a+}\frac{f(x)-f(a)}{x-a}$ provided the limit exists and $f'(b)$ is in fact $Lf'(b)=\lim_{x\to b-}\frac{f(x)-f(b)}{x-b}$, provided it exists.

If c be interior point of [a, b], then $f'(c)$ exists provided

$\lim_{x\to c-0}\frac{f(x)-f(c)}{x-c}=Lf'(c)$ exists, $\lim_{x\to c+0}\frac{f(x)-f(c)}{x-c}=Rf'(c)$ exists and

$Lf'(c)=Rf'(c)(\in\mathbb{R})$

Notes : If $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ does not exist finitely, we say that f' does not exist at c.

2. $f : D\to\mathbb{R}$ is said to be differentiable on a set $D_0 \subset D$, if the restriction of f to D_0 is differentiable at every point of D_0.

Result : 1. Let $f : D\to\mathbb{R}$ be differentiable at $p\in D\cap D'$, then there exists $\delta>0$ and a constant $M>0$ such that

$|f(x)-f(p)|\le M|x-p|$ for every $x\in D\cap N(p,\delta)$

Proof : By hypothesis $\lim_{x\to p}\dfrac{f(x)-f(p)}{x-p}$ exists & $=f'(p)$

Let $\varepsilon>0$ be given, corresponding to this ε, there exists $\delta>0$ such that $\left|\dfrac{f(x)-f(p)}{x-p}-f'(p)\right|<\varepsilon$ whenever $x\in N'(p,\delta)\cap D$

$$\Rightarrow\left|\frac{f(x)-f(p)}{x-p}\right|<\varepsilon+|f'(p)|=M(\text{say}),\ M \text{ is a positive constant,}$$

Hence $|f(x)-f(p)|<M|x-p|,\ x\in N(p,\delta)\cap D$

Note : Instead of ε in above, you can choose any fixed positive number

Corollary : If we take $\delta=\dfrac{\varepsilon}{M}$, then from above result

$|f(x)-f(p)|<M\dfrac{\varepsilon}{M}$ wherever $|x-p|<\dfrac{\varepsilon}{M}$

$\Rightarrow f$ is continuous at p

So derivability at a point $\Rightarrow$ continuity at that point

Note : converse is not true.

$$f(x)=\begin{cases} x\sin\dfrac{1}{x},\ x\neq 0\\ 0,\ x=0\end{cases}$$

As $\lim_{x\to 0}\sin\dfrac{1}{x}$ does not exist so f' does not exist at $x=0$.

But f is continuous at $x=0$.

Examples :

1. Let $f:(a,b)\to\mathbb{R}$ be differentiable at $x\in(a,b)$. Let $\{\alpha_n\}_n$ & $\{\beta_n\}_n$ be sequences such that $a<\alpha_n<x<\beta_n<b,\ \alpha_n\to x, \beta_n\to x$

Then show that $\lim_{n\to\infty}\dfrac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}=f'(x)$

Let $\lambda_n = \dfrac{\beta_n - x}{\beta_n - \alpha_n}$. Then $0 < \lambda_n < 1$

$$\frac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n} - f'(x) = \lambda_n\left\{\frac{f(\beta_n)-f(x)}{\beta_n - x} - f'(x)\right\}$$

$$+(1-\lambda_n)\left\{\frac{f(\alpha_n)-f(x)}{\alpha_n - x} - f'(x)\right\}$$

By hypothesis, $f'(x)$ exists & so the expressions within the brackets both tend to zero as $n \to \infty, \{\lambda_n\}_n$ & $\{1-\lambda_n\}_n$ are both bounded.

Hence $\lim\limits_{n\to\infty} \dfrac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}$ exists & is equal to $f'(x)$.

Note : If $x < \alpha_n < \beta_n$, the result may fail.

Let $\beta_n = \dfrac{1}{n}(n \in \mathbb{N})$ and let $\{\alpha_n\}_n$ be a sequence such that $\beta_{n+1} < \alpha_n < \beta_n$

Let $f:[-1,1] \to \mathbb{R}$ be a piecewise linear function such that $f\left(\dfrac{1}{n}\right) = \dfrac{1}{n^2}, f(\alpha_n) = 0, f(x) = 0$ for $-1 \le x \le 0$

We choose α_n nearer to β_n. Let $\alpha_n = \dfrac{1}{2}\left(\dfrac{1}{n} + \dfrac{1}{n+1}\right)$

Then $\lim\limits_{n\to\infty} \dfrac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n} = \lim\limits_{n\to\infty} \dfrac{\dfrac{1}{n^2}-0}{\dfrac{1}{n}-\dfrac{1}{2}\left(\dfrac{1}{n}+\dfrac{1}{n+1}\right)} = \dfrac{2(n+1)}{n} > 1$

But $f'(0) = 0$ & so the conclusion mentioned in the problem, fails.

2. If the function $xf(x)$ has a derivative at a given point $x_0 \neq 0$ and if $f(x)$ is continuous there, show that $f(x)$ has a derivative there.

$$\frac{xf(x)-x_0 f(x_0)}{x-x_0} - f(x) = x_0\left\{\frac{f(x)-f(x_0)}{x-x_0}\right\} \text{.......(1)}$$

By hypothesis, $xf(x)$ has a derivative at $x_0 \neq 0$ and $f(x)$ is continuous at x_0.

So as $x \to x_0$, L.H.S. of (1) $\to \frac{d}{dx}(x f(x))\Big|_{x=x_0} - f(x_0)$

Hence $\lim\limits_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$ exists and is equal to

$$\frac{1}{x_0}\left[\frac{d}{dx}(xf(x))\Big|_{x=x_0} - f(x_0)\right]$$

3. $f:(-1,1)\to\mathbb{R}$ be defined by $f(x)=\begin{cases} x^{\alpha}\sin\frac{1}{x^{\beta}}, & x\neq 0 \\ 0, & x=0\end{cases}$

Show that (i) if $\alpha=1, \beta>0$, f' does not exist at $x = 0$ but if $\alpha>1, \beta>0, f'$ exists at $x = 0$.

(ii) if $0<\beta<\alpha-1$, f' is continuous at 0

(iii) if $0<\alpha-1\leq\beta$, f' is discontinuous at 0.

$$\frac{f(x)-f(0)}{x-0} = x^{\alpha-1}\sin\frac{1}{x^{\beta}},\ x\neq 0$$

(i) if $\alpha=1, \beta>0$, $x^{\alpha-1}\sin\frac{1}{x^{\beta}} = \sin\frac{1}{x^{\beta}}$. As $\lim\limits_{x\to 0}\sin\frac{1}{x^{\beta}}(\beta>0)$ does not exist.

so in this case, f' does not exist at $x = 0$.

(ii) Let $\alpha>1, \beta>0$

Let $\varepsilon>0$ be any number.

$$\left|\frac{f(x)-f(0)}{x-0}-0\right|=\left|x^{\alpha-1}\sin\frac{1}{x^{\beta}}\right|\le|x|^{\alpha-1}<\varepsilon \text{ wherever } |x-0|<\delta=\varepsilon^{1/(\alpha-1)}$$

So $\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0}$ exists i.e. $f'(0)$ exists.

So for $0<\beta<\alpha-1, f'(0)=0$

if $x\ne 0, f'(x)=\alpha x^{\alpha-1}\sin\dfrac{1}{x^{\beta}}-\beta x^{\alpha-1-\beta}\cos\dfrac{1}{x^{\beta}}$

As $\alpha>1,\beta>0, \lim\limits_{x\to 0}x^{\alpha-1}\sin\dfrac{1}{x^{\beta}}=0$

and as $\alpha-1-\beta>0,\ \lim\limits_{x\to 0}x^{\alpha-1-\beta}\cos\dfrac{1}{x^{\beta}}=0$

Hence $\lim\limits_{x\to 0}f'(x)=f'(0)\ \&\ f'$ is continuous at $x = 0$.

(iii) Let $0<\alpha-1\le\beta$

Then $\alpha-1-\beta\le 0$, $\lim\limits_{x\to 0}x^{\alpha-1-\beta}\cos\dfrac{1}{x^{\beta}}$ does not exist

Hence f' is discontinuous at 0. (Nature of this discontinuity is that of second kind)

4. Consider a polynomial $f(x)$ with real coefficients having the property that $f[g(x)]=g[f(x)]$ for every polynomial $f(x)$ with real coefficients.

Show that $f(x) = x$

Let us take $g(x)=x+h, h\in\mathbb{R}$

So $f(x+h)=f(x)+h$ as $f[g(x)]=g[f(x)]$ by hypothesis.

$\Rightarrow\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=1\Rightarrow f(x)=x+\lambda$ where λ is real constant.

Let $g(x)=0$, then $f[g(0)]=g[f(0)]\Rightarrow 0=0+\lambda\Rightarrow f(x)=x$

5. If $f:\mathbb{R}\to\mathbb{R}$ be differentiable at $c\in\mathbb{R}$, show that

$$f'(c)=\lim_{n\to\infty}\left\{n\left[f\left(c+\frac{1}{n}\right)-f(c)\right]\right\}$$

Hence show that if f is the derivative of a function g, then f is the limit of a sequence of continuous functions.

As $f'(c)$ exists, so $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}=f'(c)$

$\lim_{n\to\infty}\frac{1}{n}=0$, so replacing h by $\frac{1}{n}$, $\lim_{n\to\infty}\frac{f\left(c+\frac{1}{n}\right)-f(c)}{1/n}=f'(c)$

So the first part follows.

By hypothesis, g is derivable & so g is continuous function.

We define $g_n(x)=g\left(x+\frac{1}{n}\right)$ & so these g_n's are continuous.

Also $n[g_n-g]$ are continuous.

By hypothesis, f is derivative of g, it follows that

$$f=\lim_{n\to\infty}n\{g_n-g\}=\lim_{n\to\infty}n\left\{g\left(x+\frac{1}{n}\right)-g(x)\right\}$$

So f is the limit of a sequence of continuous functions.

Sign of the derivative at a point.

Theorem : Let $f:I\to\mathbb{R}$ and let c be an interior point of interval I.

Let $f'(c)$ exist and $f'(c)\neq 0$

(a) If $f'(c)>0$, there exists a neighbourhood of 'c' in which f is increasing function. In other words, there exists $\delta>0$ such that

$f(x)>f(c)$ for all $x\in(c,c+\delta)\cap I$ and

$f(x)<f(c)$ for all x in $(c-\delta,c)\cap I$

(b) If $f'(c)<0$, there exists a neighbourhood of '*c*' in which *f* is decreasing function. In other words, there exists $\delta>0$ such that

$f(x)<f(c)$ for all $x\in(c,c+\delta)\cap I$

$f(x)>f(c)$ for all $x\in(c-\delta,c)\cap I$

Proof : Let $0<\varepsilon<\dfrac{|f'(c)|}{2}$ be any number.

Corresponding to such ε, there exists $\delta>0$ such that

$$\left|\frac{f(x)-f(c)}{x-c}-f'(c)\right|<\frac{|f'(c)|}{2} \text{ whenever } 0<|x-c|<\delta(x\in I)$$

Case I : Let $f'(c)>0$. Then $\dfrac{f'(c)}{2}<\dfrac{f(x)-f(c)}{x-c}<\dfrac{3f'(c)}{2}$

whenever $c-\delta<x<c,\ c<x<c+\delta$

If $c-\delta<x<c,\ f(x)-f(c)<\dfrac{(x-c)f'(c)}{2}<0\Rightarrow f(x)<f(c)$ in $c-\delta<x<c$

If $c<x<c+\delta, f(x)-f(c)>\dfrac{(x-c)f'(c)}{2}>0\Rightarrow f(x)>f(c)$ in $c<x<c+\delta$

Consequently, *f* is increasing in the neighbourhood of *c*.

Case II : Let $f'(c)<0$. Then $\varepsilon=-\dfrac{f'(c)}{2}$, so we have

$$\frac{3f'(c)}{2}<\frac{f(x)-f(c)}{x-c}<\frac{f'(c)}{2}, 0<|x-c|<\delta.$$

If $c-\delta<x<c, f(x)-f(c)>\dfrac{(x-c)f'(c)}{2}>0$

$\Rightarrow f(x)>f(c)$ in $c-\delta<x<c$

If $c < x < c+\delta, f(x)-f(c) < \dfrac{3(x-c)f'(c)}{2} < 0$

$\Rightarrow f(x) < f(c)$ in $c < x < c+\delta$

So f is decreasing in the $\delta-$ neighbourhood of c.

Note : If $f'(c)=0$, no conclusion can be drawn.

Let $f(x)=x^3$, then $f'(x)=3x^2$ and $f'(0)=0$

In $0 < x < 0+\delta$ & in $0-\delta < x < 0, f(x)-f(0) < 0$

So f is increasing in $N'(0,\delta)\cap D_f$

Let $f(x)=x^2$. Then $f'(x)=2x$ & $f'(0)=0$

$f(x)-f(0) > 0$ in both $0-\delta < x < 0$ & $0 < x < 0+\delta$

$f(x)$ is neither increasing nor decreasing in any δ-neighbourhood of 0.

3.3 Algebra of differentiable functions

Let f and g are two functions differentiable at $c(\in D_f \cap D_g)$, then

(i) $\alpha f(x)$ is differentiable at c where $\alpha \in \mathbb{R}$

(ii) $f \pm g$ are differentiable at c & $(f \pm g)'(c) = f'(c) \pm g'(c)$

(iii) $f\,g$ is differentiable at c & $(fg)'(c) = f(c)g'(c) + f'(c)g(c)$

(iv) if $g(c) \neq 0, \dfrac{f}{g}$ is differentiable at c and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c)-f(c)g'(c)}{\{g(c)\}^2}$$

(v) $|f|$ is differentiable at c, $f(c) \neq 0$

Proof : Deduction of (i) and (ii) are simple and follow straight way from the definition of derivative.

(iii) $\dfrac{(fg)(x)-(fg)c}{x-c}=f(x)\left\{\dfrac{g(x)-g(c)}{x-c}\right\}+g(c)\left\{\dfrac{f(x)-f(c)}{x-c}\right\}$

Existence of f', g' at $c \Rightarrow \lim\limits_{x\to c}\dfrac{f(x)-f(c)}{x-c}=f'(c)$, $\lim\limits_{x\to c}\dfrac{g(x)-g(c)}{x-c}=g'(c)$

Due to continuity of *f*, *g* at *c*,

$$\lim_{x\to c}\frac{(fg)(x)-(fg)(c)}{x-c}=f(c)g'(c)+g(c)f'(c)$$

(iv) Given $g(c)\neq 0$, due to continuity of g at c, there exists *nbd* of *c*

or, interval *I* having c as its interior point such that $g(x)\neq 0$ in *I*

Let $x\in D\left(\dfrac{f}{g}\right)\cap I$

$$\frac{\left(\frac{f}{g}\right)(x)-\left(\frac{f}{g}\right)c}{x-c}=\frac{1}{g(x)g(c)}\left\{g(c).\frac{f(x)-f(c)}{x-c}-f(c).\frac{g(x)-g(c)}{x-c}\right\}$$

By hypothesis, $\lim\limits_{x\to c}\dfrac{f(x)-f(c)}{x-c}=f'(c), \lim\limits_{x\to c}\dfrac{g(x)-g(c)}{x-c}=g'(c)$

Due to continuity of g at c, $\lim\limits_{x\to c}g(x)=g(c)$

So $\lim\limits_{x\to c}\dfrac{\left(\frac{f}{g}\right)(x)-\left(\frac{f}{g}\right)(c)}{x-c}=\dfrac{1}{\{g(c)\}^2}\left[f'(c)g(c)-f(c)g'(c)\right]$

(v) $\left|\dfrac{|f|(x)-|f|(c)}{x-c}\right|\leq\left|\dfrac{f(x)-f(c)}{x-c}\right|$

As the limit of RHS as $x \to c$ exists finitely, so the limit $\lim_{x \to c} \frac{|f|(x) - |f|(c)}{x - c}$ exists finitely.

Note The condition $f(c) \neq 0$ is required. otherwise the result may fail.

For example, $f(x) = x$ and $c = 0$.

Derivative of composite function (Chain Rule)

Theorem : If f is differentiable at c and g is differentiable at $f(c)$, then the composite function $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c)).f'(c)$$

Note that c is interior point of domain of $g \circ f$.

Let us consider the function $h : D_g \to \mathbb{R}$ as follows :

$$h(y) = \begin{cases} \dfrac{g(y) - g(f(c))}{y - f(c)}, & y \neq f(c) \\ g'(f(c)), & y = f(c) \end{cases}$$

Then $\lim_{y \to f(c)} h(y) = g'(f(c)) = h(f(c))$ & so h is continuous at $f(c)$.

Again $g(y) - g(f(c)) = (y - f(c))h(y)$ for all $y \in D_g$ (by construction of h)

Hence for $x \in D_{g \circ f}$

$$(g \circ f)(x) - (g \circ f)(c) = h(f(x))(f(x) - f(c))$$

$\Rightarrow$ for $x \in D_{g \circ f}$ & $x \neq c$

$$\frac{(g \circ f)(x) - (g \circ f)(c)}{x - c} = h(f(x)).\frac{f(x) - f(c)}{x - c}$$

Continuity of f at c & continuity of h at $f(c) \Rightarrow$ $h \circ f$ is continuous at c.

As $x \to c, RHS \to h(f(c))$

So $\lim_{x\to c}\frac{(g\circ f)(x)-(g\circ f)(c)}{x-c}$ exists and is $h(f(c)).f'(c)$

$\Rightarrow g\circ f$ is differentiable at c and

$(g\circ f)'(c)=g'(f(c)).f'(c)$

Derivative of inverse function

Let f be strictly monotone and continuous in an interval $I(\subset\mathbb{R})$

and let x_0 be an interior point of I at which f has a derivative $f'(x_0)\neq 0$.

Then f^{-1} has a derivative at this point $y_0=f(x_0)$, equal to $1/f'(x_0)$

Proof : Here domain of f^{-1} is an interval J (say).

By hypothesis, x_0 is an interior point of I. By definition of interior point, there exists points $p,q\in I$ such that $p<x_0<q$ and then $f(x_0)$ is interior point of the closed interval J_1 = $[f(p), f(q)]$ as f is strictly monotone.

f is continuous on $[p,q]\subset I$, the interval $J_1\subset J$ (by I.V. property of continuous function), so $y_0=f(x_0)$ is interior point of J.

Now $\lim_{y\to y_0}\left\{\frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0}\right\}=\lim_{y\to y_0}\left\{\frac{f^{-1}(y)-f^{-1}(y_0)}{f(f^{-1}(y))-f(f^{-1}(y_0))}\right\}.$

Due to continuity of f^{-1}, f^{-1} is continuous at y_0, so that

$\lim_{y\to y_0}f^{-1}(y)=f^{-1}(y_0)=x_0$

Following substitution rule for composite function

$\lim_{y\to y_0}\left\{\frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0}\right\}=\lim_{x\to x_0}\left\{\frac{x-x_0}{f(x)-f(x_0)}\right\}$

since $f'(x_0)\neq 0$, we get

$$\lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{\lim\limits_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)}$$

consequenctly, $\lim\limits_{y \to y_0} \left\{ \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} \right\} = \frac{1}{f'(x_0)}$

Note : Alternative proofs of the last two results follow from concept of differentiability, discussed subsequently.

Diffentiability and differential

$f(x)$ is said to be differentiable at a point of its domain if

$f(x + \Delta x) - f(x) = A.\Delta x + \varepsilon.\Delta x$ where A is independent of Δx and $\varepsilon \to 0$ as $\Delta x \to 0$.

Let f be differentiable at x

From above definition, $\frac{f(x + \Delta x) - f(x)}{\Delta x} = A + \varepsilon$

Taking $\Delta x \to 0$, $RHS \to A$ so $\lim\limits_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ exists & $(= f'(x)) = A$

So differentiabity at a point of its domain $\Rightarrow$ existence of first order derivative at that point.

Converse let $f'(x)$ exist. so $\lim\limits_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ exists & $= f'(x)$

Let $\frac{f(x + \Delta x) - f(x)}{\Delta x} - f'(x) = \varepsilon$ & so $\varepsilon \to 0$ as $\Delta x \to 0$

$\Rightarrow f(x + \Delta x) - f(x) = f'(x)\Delta x + \varepsilon.\Delta x$ where $\varepsilon \to 0$ as $\Delta x \to 0$

$\Rightarrow f$ is differentiable at x and hence differentiability $\Leftrightarrow$ existence of derivative at that point.

Note : 1. This result is of importance in the sense that the result differes for functions $f : \mathbb{R}^2 \to \mathbb{R}$

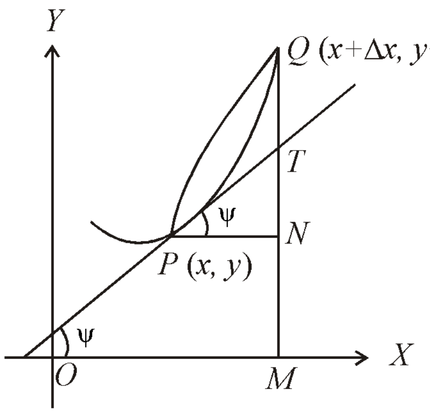
2. If $y = f(x)$ & $y + \Delta y = f(x + \Delta x)$, then

$\Delta y = f(x + \Delta x) - f(x) = f'(x).\Delta x + \varepsilon.\Delta x$ where $\varepsilon \to 0$ as $\Delta x \to 0$

Δy is increment of y for the increment Δx of x.

$f'(x).\ \Delta x$ is known as the differential of y, denoted by dy.

This $dy \neq \Delta y$ but $dx = \Delta x$ (taking $f(x) = x$, it is evident)



$P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ are two neighbouring points on the curve.

$$\tan\psi = \frac{dy}{dx} = f'(x) = \frac{TN}{PN} \Rightarrow TN = f'(x)\Delta x$$

$\Rightarrow dy = TN$

So $dy = TN$ but $\Delta y = PN$

& So $dy \neq \Delta y$

Alternative proof for diffentiability of composite function and Chain Rule.

Let f be differentiable at $x(\in D_f)$ & g be differentiable at $f(x)(\in D_g)$.

Here we assume that the composite function $g \circ f$ can be defined in the sense that $(g \circ f)(x) = g(f(x)),\ x \in D_f$.

Here $\Delta y = f(x + \Delta x) - f(x) = f'(x)\Delta x + \varepsilon.\Delta x$ where $\varepsilon \to 0$ as $\Delta x \to 0$ (taking $y = f(x)$.......(1)

Taking $x = g(t)$

$\Delta x = g(t+\Delta t) - g(t) = g'(t)\Delta t + \eta.\ \Delta t$ where $\eta \to 0$ as $\Delta t \to 0$ ....(1)

$\Delta y = \left(f'(x)+\varepsilon\right)\left(g'(t)\Delta t + \eta\Delta t\right)$ where $\eta \to 0$ as $\Delta t \to 0$ ....(2)

$\Delta y = \left(f'(x)+\varepsilon\right)\left(g'(t)\Delta t + \eta\Delta t\right)$

$= f'(x)g'(t)\Delta t + \left(f'(x).\eta + \varepsilon.g'(t) + \varepsilon\eta\right)\Delta t$ (3)

As $\Delta t \to 0, \Rightarrow \Delta x \to 0$ due to the continuity of g.

As $\Delta x \to 0, \varepsilon \to 0$. Consequently, $f'(x).\eta + \varepsilon.\ j'(t) + \varepsilon\eta \to 0$ as $\Delta t \to 0$

Recalling (3), $\Delta y = f'(x)g'(t)dt + \tau.\Delta t\left(\tau = f'(x).\ \eta + \varepsilon.\ g'(t) + \varepsilon\eta\right)$

where $\tau \to 0$ as $\Delta t \to 0$

$\Rightarrow y$ is a differentiable function of t and $\dfrac{dy}{dt} = \dfrac{dy}{dx}.\dfrac{dx}{dt}$

Note Similarly the differentiability of inverse function can be discussed.

Theorem (Darboux Theorem) : Let $f:[a,b] \to \mathbb{R}$ be derivable in the closed and bounded interval $[a, b]$ and $f'(a)f'(b) < 0$. Then there exists at least one point $c \in (a,b)$ such that $f'(c) = 0$.

Proof : Let $f'(a) > 0,\ f'(b) < 0$

f, being derivable in $[a, b]$ is continuous in $[a, b]$. So f is bounded in $[a, b]$ and attains its bounds in $[a, b]$. So there are points $\mathrm{c, d} \in [\mathrm{a,b}]$ such that $\sup\limits_{[a,b]} f = M = f(c)$ and $\inf\limits_{[a,b]} f = m = f(d)$.

As $f'(a) > 0$, So $f(x)$ is increasing in some neighbourhood of a and hence there exists $\delta > 0$ that $f(x) > f(a)$ in $a < x < a + \delta$.

If $c = a$, then $f(x) > M$ in $a < x < a + \delta$ which is absurd. So $c \neq a$.

As $f'(b) < 0,\ f(x)$ is decreasing in some neighbourhood of b & so there exists $\eta > 0$ such that $f(x) > f(b)$ in $b - \eta < x < b$.

If $c = b$, then $f(x) > M$ in $b - \eta < x < b$ which is also absurd. So $c \neq b$.

So $c \in (a, b)$. By hypothesis, $f'(c)$ exists. We propose to show that $f'(c) = 0$.

If possible, let $f'(c) > 0$. Then there exists $\delta' > 0$ such that $f(x) > f(c)(= M)$ in $(c, c + \delta')\ (\subset [a, b])$ this is absurd, so $f'(c) \not> 0$.

If possible, let $f'(c) < 0$. Then there exists $\eta' > 0$ such that $f(x) > f(c)(= M)$ in $(c - \eta', c)(\subset [a, b])$ & this is absurd, so $f'(c) \not< 0$. Hence $f'(c) = 0$.

Corollaries (1) : Let $f : [a, b] \to \mathbb{R}$ be derivable in $[a, b]$ & $f'(a) \neq f'(b)$. If k be any number between $f'(a)$ and $f'(b)$, then there exists at least one point $c \in (a, b)$ such that $f'(c) = k$.

Let us construct $\varphi : [a, b] \to \mathbb{R}$ defined by $\varphi(x) = f(x) - kx$.

Derivability of f in $[a, b] \Rightarrow$ derivability of φ in $[a, b], \varphi'(x) = f'(x) - k$

$\varphi'(a)\varphi'(b) = (f'(a) - k)(f'(b) - k) < 0$

So by Darboux Theorem, there exists $c \in (a, b)$ for which $\phi'(c) = 0$ i.e. $f'(c) = k$.

(2) If f be derivable in a closed and bounded interval I, then the range set of f' on I is either a singleton or an interval.

If two distinct members $p_1, p_2 \in J$, there exists distinct elements $x_1, x_2 \in I$ such that $f'(x_i) = p_i$ for i = 1, 2. Let $x_1 < x_2$. So $[x_1, x_2] \subset I$.

If $p_1 < p < p_2$, by Darboux's theorem on derivative, there exists $c \in (x_1, x_2)$

such that $f'(c) = p$. So $p \in J$. But p is arbitrary point between p_1 & p_2. This shows that if $p_1, p_2, \in J$, then every element between p_1 & p_2 belongs to J. So J is an interval in $\mathbb{R}$.

(3) Let $f:[a, b]\to\mathbb{R}$ be derivable on $[a, b]$. Then f' can not have any jump discontinuity on $[a, b]$

Let $c\in(a,b)$. We propose to show that

(i) if $\lim\limits_{x\to c-} f'(x)$ exists, then it is $f'(c)$, $c\in(a,b]$

(ii) if $\lim\limits_{x\to d+} f'(x)$ exists, then it is $f'(d)$, $d\in[a,b)$

(i) Let $a<c\le b$ & $\lim\limits_{x\to c-} f'(x)=l(\in\mathbb{R})$. We have to show that $l=f'(c)$

Let $l<f'(c)$. Let $0<\varepsilon<f'(c)-l$.

As $\lim\limits_{x\to c-0} f'(x)=l$, corresponding to above chosen $\varepsilon>0$, there exists $\delta>0$ such that $|f'(x)-l|<\varepsilon$ whenever $x\in(c-\delta,c)\cap[a,b]$

So if $p\in(c-\delta,c)\cap[a,b]$, then $l-\varepsilon<f'(p)<l+\varepsilon<f'(c)$ (by above ε)

So by Darboux theorem, there exists point $\xi\in(p, c)$ such that $f'(\xi)=l+\varepsilon$.

Now $\xi\in(p,c)\Rightarrow\xi\in(c-\delta,c)\cap[a,b]$ & so by above $f'(\xi)<l+\varepsilon$. Thus we arrive at a contradiction. So $l\not<f'(c)$.

If possible let $l>f'(c)$. We choose ε such that $0<\varepsilon<l-f'(c)$

$\lim\limits_{x\to c-0} f'(x)=l\Rightarrow$ Corresponding to above ε, there exists $\eta>0$ such that

$l-\varepsilon<f'(x)<l+\varepsilon$ whenever $x\in(c-\eta,c)\cap[a,b]$.

Let $q\in(c-\eta,c)\cap[a,b]$. Then $f'(c)<l-\varepsilon<f'(q)$.

Again by Darboux theorem on derivative. there exists point τ in (q,c)

such that $f'(\tau)=l-\varepsilon$.

But $\tau\in(q,c)\Rightarrow\tau\in(c-\eta,c)\cap[a,b]$ and hence $f'(\tau)>l-\varepsilon$.

We arrive at a contradiction. So $l \not> f'(c)$.

As a result $\lim\limits_{x \to c-0} f'(x) = f'(c),\ a < c < b$

Similarly it can be shown that $\lim\limits_{x \to c+0} f'(x)$ exists $= f'(c)$

So a derived function on an interval $[a, b](\subset \mathbb{R})$ can have a discontinuity of second kind only

(3) Let $f'(x)$ exist and be monotone on an open interval (a, b). Then f' is continuous on (a, b).

If possible, let f' have a discontinuity at some point $c \in (a, b)$.

c is interior point of (a, b) & we have a closed sub interval $[\alpha, \beta]$ of (a, b) which contains c in its interior.

By hypothesis f' is monotone in $[\alpha, \beta]$ & so the discontinuity at c must be a jump discontinuity. But a derived function can not have any jump discontinuity . So f' is continuous on (a, b).

3.4 Theorem (Rolle's theorem)

Let $f : [a, b] \to \mathbb{R}$ be

(i) continuous in $[a, b]$ (ii) derivable in (a, b) (iii) $f(a) = f(b)$.

Then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.

Proof : Continuity of f in $[a, b]$ ensures the boundedness of f in $[a, b]$ & attainment of bounds in $[a, b]$. Let $M = \sup\limits_{[a,b]} f$ and $m = \inf\limits_{[a,b]} f$.

There are points $c, d \in [a, b]$ such that $f(c) = M$, $f(d) = m$.

Case I : Let $M = m$. Then $f(x)$ is constant function in $[a, b]$ & so $f'(x) = 0$ in (a, b).

Case II : Let $M \neq m$. As $f(a) = f(b)$, So at least one of M and m is different from $f(a)$ and $f(b)$. So $c \neq a, c \neq b$ (if M be different from $f(a)$ and $f(b)$)

Hence $c \in (a, b)$ & by hypothesis (ii) $f'(c)$ exists.

If $f'(c) > 0$, there exists $\delta > 0$ such that $f(x) > f(c)$ in $c < x < c + \delta$ where $(c, c+\delta) \subset (a, b)$. So $f(x) > M$ in $(c, c+\delta)$ which is absurd. So $f'(c) \not> 0$.

If $f'(c) < 0$, then there exists $\eta > 0$ such that $f(x) > f(c)$ in $(c - \eta, c) \subset (a, b)$.

Again $f(x) > M$ in $(c - \eta, c)$ which is absurd. thus $f'(c) \not< 0$.

Consequence $f'(c) = 0$.

Note : The above theorem gives a set of sufficient conditions for the vanishing of f' at an interior point of D_f. The conditions are not necessary. For example,

$$f(x) = \frac{1}{x-1} + \frac{1}{2-x}, \quad 1 < x < 2$$

$f'(x) = 0$ at $x = \frac{3}{2}$ but f does not obey the conditions of Rolle's theorem in [1, 2].

Geometrical interpretation of Rolle's Theorem

If the two end points of the graph of $y = f(x)$ be on the same horizontal line (i.e. on a line parallel to x-axis) and if the graph be continuous throughout the interval and if the curve has a tangent at every point on it except possibly the two end points, then there must exist at least one point on the curve at which the tangent is parallel to x-axis.

Examples :

(1) Let $f, g, h : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$ and be derivable in (a, b), then show that there exists $c \in (a, b)$ for which

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$$

Let us construct $F:[a,b]\to\mathbb{R}$ as $F(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$

Continuity of *f*, *g*, *h* in $[a,b]\Rightarrow$ continuity of *F* in [*a*, *b*].

$F'(x)=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$ exists in (*a*, *b*), as f', g', h' exist in (*a*, *b*)

Also *F*(*a*) = 0 = *F*(*b*). So *F*(*x*) satisfies all the conditions of Rolle's Theorem in [*a*, *b*]. Therefore by Rolle's theorem, there exists $c\in(a,b)$ s.t. $F'(c)=0$

Hence the result follows.

(2) Let *f*, *g* be differentiable on the inerval *I*. Let $a,b\in I$ and $a<b$ and

f(a) = 0 = *f*(*b*). Show that there exists $c\in(a,b)$ such that

$f'(c)+f(c)g'(c)=0$

We construct the function $h:[a,b]\to\mathbb{R}$ as $h(x)=f(x).e^{g(x)}$

Continuity of *f* & *g* in $[a,b]\Rightarrow$ continuity of *h* in [*a*, *b*].

$h'(x)=f'(x)e^{g(x)}+f(x)e^{g(x)}.g'(x)$ exists in (*a*, *b*) as *f*, *g*, are derivable in (*a*, *b*)

$h(a)=0=h(b)$ by given condition. So by Rolle's theorem, there exists

$c\in(a,b)$ such that $h'(c)=0\Rightarrow e^{g(c)}\{f'(c)+f(c)g'(c)\}=0$.

As $e^{g(c)}\neq 0$, so $f'(c)+f(c)g'(c)=0$ for some $c\in(a,b)$

Particular Case $f'(c)+\lambda f(c)=0\,(\lambda\in\mathbb{R})$ under the same set of conditions mentioned above.

(3) $x^4+2x^2-6x+2=0$ has

(A) 4 real roots (B) exactly two real roots (C) no real root (D) one pair of equal roots.

$f(0) = 2, f(1) = -1, f(2) = 14$. $f(x)$ is continuous function &

$f(0)f(1) < 0, \quad f(1)f(2) < 0$.

By Bolzano's theorem on continuous function, $f(x)$ must vanish at least once in (0, 1) & at least once in (1, 2).

If possible, let it have more than two real roots. Then by Rolle's Theorem, $f'(x)$ must vanish at least twice & $f''(x)$ must vanish at least once. But $f''(x) = 12x^2 + 4 > 0$ for all x. (B) is true.

(4) If $a < c < b$ and $f''(x)$ exists finitely in $[a, b]$, then there exists $k \in (a, b)$ such that

$$\frac{f(a)}{(a-b)(a-c)} + \frac{f(b)}{(b-c)(b-a)} + \frac{f(c)}{(c-a)(c-b)} = \frac{1}{2} f''(k)$$

Let us construct the function $\phi : [a, b] \to \mathbb{R}$ as follows.

$$\phi(x) = \frac{(x-b)(x-c)}{(a-b)(a-c)} f(a) + \frac{(x-c)(x-a)}{(b-c)(b-a)} f(b) + \frac{(x-a)(x-b)}{(c-a)(c-b)} f(c) - f(x)$$

Continuity of f in $[a, b] \Rightarrow$ Continuity of ϕ in $[a, b]$.

$$\phi'(x) = \frac{2x-(b+c)}{(a-b)(a-c)} f(a) + \frac{2x-(a+c)}{(b-c)(b-a)} f(b) + \frac{2x-(a+b)}{(c-a)(c-b)} f(c) - f'(x)$$

& as $f'(x)$ exists in (a, b), ϕ' exists in (a, b). Also $\phi(a) = \phi(b) = \phi(c) = 0$

Given that $a < c < b$, so ϕ' satisfies the conditions of Rolle's theorem in both $[a, c]$ & $[c, b]$.

By Rolle's theorem, there exists $\xi_1 \in (a, c)$ & $\xi_2 \in (c, b)$ such that

$\phi'(\xi_1) = 0 = \phi'(\xi_2)$

As $f''(x)$ exists in (a, b)

$$\varphi''(x)=\frac{2}{(a-b)(a-c)}f(a)+\frac{2}{(b-c)(b-a)}f(b)+\frac{2}{(c-a)(c-b)}f(c)-f''(x)$$

exists in (ξ_1,ξ_2). Applying Rolle's theorem to ϕ' in $[\xi_1,\xi_2]$, there exists $k\in(\xi_1,\xi_2)\subset(a,b)$ such that $\phi''(k)=0$

$$\Rightarrow\frac{f(a)}{(a-b)(a-c)}+\frac{f(b)}{(b-c)(b-a)}+\frac{f(c)}{(c-a)(c-b)}=\frac{1}{2}f''(k).$$

(5) let $f,g:[a,b]\to\mathbb{R}$ be such that each is derivable in (a, b), each is continuous at a & b. Then there exists $c\in(a,b)$ such that

$$f'(c)\{g(b)-g(a)\}=g'(c)\{f(b)-f(a)\}$$

We construct $h:[a,b]\to\mathbb{R}$ as follows :

$h(x)=f(x)\{g(b)-g(a)\}-g(x)\{f(b)-f(a)\}$ for all $x\in[a,b]$

Continuity of f & g in $[a,b]\Rightarrow$ Continuity of h in [a, b].

$h'(x)=f'(x)\{g(b)-g(a)\}-g'(x)\{f(b)-f(a)\}$ exists in (a, b) as f',g' exist in (a, b).

$h(a)=f(a)g(b)-f(b)g(a)$, $h(b)=-f(b)g(a)+f(a)g(b)$

and so $h(a)=h(b)$

So h satisfies all the conditions of Rolle's theorem in [a, b]. By Rolle's theorem, there exists $c\in(a,b)$ such that $h'(c)=0$.

$$\Rightarrow f'(c)\{g(b)-g(a)\}=g'(c)\{f(b)-f(a)\}$$

(6) Show that between any two real roots of $e^x\sin x=1$, there is at least one real root of $e^x\cos x+1=0$.

Let $f(x)=e^x\sin x-1$ and a, b be two real roots of $f(x)=0$

Let $g(x) = e^{-x} - \sin x, \quad a \le x \le b$.

g is continuous in $[a, b]$ and $g'(x) = -e^{-x} - \cos x$ exists in (a, b)

Also $g(a) = g(b) = 0$ by above hypothesis.

g satisfies all the conditions of Rolle's theorem in $[a, b]$ & so by Rolle's theorem, there exists $c \in (a, b)$ such that $g'(c) = 0$ i. e. $e^{-c} + \cos c = 0$ or $1 + e^{c} \cos c = 0$.

$\Rightarrow c$ is root of $e^{x} \cos x + 1 = 0$

(7) Let $f : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$, differentiable in (a, b) and be nowhere zero in (a, b). Show that there exists $\theta \in (a, b)$ such that

$$\frac{f'(\theta)}{f(\theta)} = \frac{1}{a-\theta} + \frac{1}{b-\theta}$$

We construct $g : [a, b] \to \mathbb{R}$ as follows :

$g(x) = (x-a)(x-b) f(x)$ for all $x \in [a, b]$

Continuity of f in $[a, b] \Rightarrow$ continuity of g in $[a, b]$

$g'(x) = (x-b) f(x) + (x-a) f(x) + (x-a)(x-b) f'(x)$ exists in (a, b) as f' exists in (a, b). Also $g(a) = 0 = g(b)$. Applying Rolle's theorem to g in $[a, b]$, there exists at least one $\theta \in (a,b)$ such that $g'(\theta) = 0$.

$$\Rightarrow (\theta - b) f(\theta) + (\theta - a) f(\theta) + (\theta - a)(\theta - b) f'(\theta) = 0$$

$$\Rightarrow \frac{f'(\theta)}{f(\theta)} = \frac{1}{a-\theta} + \frac{1}{b-\theta}$$

(8) Show that the equation $x \log x = 3 - x$ has at least one root in $(1, 3)$.

Let $f : [1, 3] \to \mathbb{R}$ be defined as follows : $f(x) = (x-3)\log x$.

f is continuous in $[1, 3]$, $f'(x) = \log x + \dfrac{x-3}{x}$ exists in $(1, 3)$ & $f(1) = 0 = f(3)$

Applying Rolle's theorem to f in $[1, 3]$, there exists $c \in (1, 3)$ such that $f'(c) = 0$

$\Rightarrow \log c + 1 - \dfrac{3}{c} = 0$ i.e. c is root of $x \log x + x = 3$

(9) If f', g' exist in $[a, b]$ & $g'(x) \neq 0$ in (a, b), show that there exists $c \in (a, b)$

such that $\dfrac{f(c) - f(a)}{g(b) - g(c)} = \dfrac{f'(c)}{g'(c)}$

We construct $h : [a, b] \to \mathbb{R}$ as follows :

$h(x) = f(x)g(x) - f(a)g(x) - g(b)f(x)$ for all $x \in [a, b]$

Existence of f', g' in $[a, b] \Rightarrow$ continuity & derivability of h in $[a, b]$

Also $h(a) = -g(b)f(a) = h(b)$. Applying Rolle's theorem to h in $[a, b]$, there exists $c \in (a, b)$ such that

$h'(c) = 0 \Rightarrow f'(c)g(c) + f(c)g'(c) - f(a)g'(c) - g(b)f'(c) = 0$

$\Rightarrow \dfrac{f'(c)}{g'(c)} = \dfrac{f(c) - f(a)}{-g(c) + g(b)}$

EXERCISE

1. If f', g' are continuous in $[a-h,\ a+h]$, derivable in $(a-h,\ a+h)$, $g''(x) \neq 0$, show that there exists $d \in (a-h, a+h)$ such that

$$\frac{f(a+h) - 2f(a) + f(a-h)}{g(a+h) - 2g(a) + g(a-h)} = \frac{f''(d)}{g''(d)}$$

2. Let $f, g:[a, b] \to \mathbb{R}$ be continuous in $[a, b]$. Assume that g, g' are nowhere zero in $[a, b]$ & (a, b) respectively..

Let $\frac{f(a)}{g(a)} = \frac{f(b)}{g(b)}$. Show that there exists $c \in (a, b)$ such that $\frac{f(c)}{g(c)} = \frac{f'(c)}{g'(c)}$

3. Let $\sum_{k=0}^{n} \frac{C_k}{k+1} = 0$ where $C_k \in \mathbb{R}$ for all k. Show that the equation

$C_0 + C_1 x + \ldots\ldots\ldots + C_n x^n = 0$ has at least one root in $(0, 1)$

4. Let $u(x), v(x), u'(x), v'(x)$ are all continuous on $\mathbb{R}$ and $uv' - u'v \neq 0$ in $\mathbb{R}$. Prove that between any two real roots of $u(x) = 0$, there lies one root of $v(x) = 0$.

5. Examine whether the equation $x^3 - 3x + k = 0, k \in \mathbb{R}$, has two distinct roots in $(0, 1)$.

6. **Correct or justify the statement :** Rolle's Theorem is not applicable to $|x|$ in any interval $[a, b] \subset \mathbb{R}$.

7. **Using Rolle's theorem**, show that the derivative $f'(x)$ of the function

$$f(x) = \begin{cases} x \sin \frac{\pi}{x}, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \end{cases}$$

vanishes on an infinite set of points of the interval $(0, 1)$.

8. Let $f, g:[a, b] \to \mathbb{R}$ be continuous in $[a, b]$, f'', g'' exist in (a, b), f & g vanish at end points a and b, $g''(x) \neq 0$ in (a, b).

If $a < c < b$ & $g(c) \neq 0$, show that there exists $\xi \in (a, b)$ such that

$$\frac{f(c)}{g(c)} = \frac{f''(\xi)}{g''(\xi)}$$

(Hints : Contruct $F:[a,b]\to\mathbb{R}$ as $F(x)=f(c)g(x)-g(c)f(x)$ for all $x\in[a,b]$)

9. let $f:\mathbb{R}\to\mathbb{R}$ be differentiable upto any number of times and let

for some $n\in\mathbb{N}$, $f(0)=f'(0)=...=f^{n}=0$

Show that $f^{(n+1)}(x)=0$ for some $x\in(0,1)$

10. Show that each of the equations

(i) $\sin(\cos x)=x$ (ii) $\cos(\sin x)=x$ has exactly one root in $\left(0,\pi/2\right)$

Lagrange's Mean value theroem or first mean value theorem of Differential calculus.

Let $f:[a,b]\to\mathbb{R}$ be (i) continuous in [*a*, *b*] (ii) derivable in (*a*, *b*). Then there exists at least one point $c\in(a,b)$ such that $f(b)-f(a)=(b-a)f'(c)$.

Proof : We construct $F:[a,b]\to\mathbb{R}$ as follows.

$F(x)=f(x)+Ax$ where the constant A is to be determined from the functional relation $F(b)=F(a)$

Continuity of *f* in $[a,b]\Rightarrow$ continuity of F in $[a,b]$ &

derivability of *f* in $(a,b)\Rightarrow$ derivability of F in (*a*, *b*). By construction, $F(b)=F(a)$

So F satisfies all the conditions of Rolle's theorem in $[a,b]$. Hence by Rolle's theorem, there exists $c\in(a,b)$ such that $F'(c)=0$.

So $F'(c)=f'(c)+A=0\to A=-f'(c)\;\&\;F(b)=F(a)\Rightarrow -A=\dfrac{f(b)-f(a)}{b-a}$

Therefore $\dfrac{f(b)-f(a)}{b-a}=f'(c)$

Note : 1. Conditions stated above are sufficient but not necessary.

Consider the function $f:[0,3]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}0, & 0\le x<\frac{1}{2}\\ x+\frac{1}{2}, & \frac{1}{2}\le x\le\frac{3}{2}\\ \frac{2}{3}x+1, & \frac{3}{2}<x\le 3\end{cases}$$

f does not sastisfy the conditions stated above but

$$\frac{f(3)-f(0)}{3-0}=1=f'\left(\frac{3}{4}\right)$$

2. Geometrical Interpretation : If the graph of a function be a continuous curve having tangent at every point on it except possibly the two end points, then there is at least one point on the curve at which the tangent is parallel to the chord joining the end points.

Examples : (i) Let f have the property that $|f'(x)|<1$ for all x in $(0,1)$ and let f be continuous at $x=0,1$. Show tht the sequence $\left\{f\left(\frac{1}{n}\right)\right\}_n$ is convergent.

We note that LMV theorem is applicable to f in any interval $\subset\mathbb{R}$

Let $\varepsilon>0$ be given. By Archimedean property of $\mathbb{R}$, there exists natural number K such that $K\varepsilon>2$. Let $m,n\in\mathbb{N}$ be such that $m,n>K$.

By L M V theorem, there exists at least one point $c\in\left(\frac{1}{m},\frac{1}{n}\right)\left(\text{or}, c\in\left(\frac{1}{n},\frac{1}{m}\right)\right)$

Such that $\left|f\left(\frac{1}{m}\right)-f\left(\frac{1}{n}\right)\right|=\left|\frac{1}{m}-\frac{1}{n}\right||f'(c)|<\left|\frac{1}{m}-\frac{1}{n}\right|$ (by hypothesis)

$$\Rightarrow\left|f\left(\frac{1}{m}\right)-f\left(\frac{1}{n}\right)\right|<\frac{2}{K}<\varepsilon \text{ for } m,n,>K$$

$\Rightarrow \left\{ f\left(\frac{1}{n}\right) \right\}_n$ is cauchy sequence in $\mathbb{R}$ & so $\left\{ f\left(\frac{1}{n}\right) \right\}_n$ is convergent sequence in $\mathbb{R}$. Hence $\left\{ f\left(\frac{1}{n}\right) \right\}_n$ has a limit in $\mathbb{R}$

2. Find real solutions of $2^x + 5^x = 3^x + 4^x$

The equation can be written as $5^x - 4^x = 3^x - 2^x$

We consider the function $f(t) = t^x$ in (i) [4, 5] (ii) [2, 3]

f is continuous in both the intervals & is derivable in both, Applying L M V theorem to t^x in both [4, 5] and [2, 3], we see that there are points $t_1 \in (4, 5)$ & $t_2 \in (2, 3)$ so that

$5^x - 4^x = xt_1^{x-1}$ & $3^x - 2^x = xt_2^{x-1}$

Therefore $xt_1^{x-1} = xt_2^{x-1} \Rightarrow \left(\frac{t_1}{t_2}\right)^{x-1} = 1 \Rightarrow x - 1 = 0$ as t_1, t_2 belong to different sub intervals & so $t_1 \neq t_2$

Hence, $x = 1$

Hence $x = 0, 1$ are only solutions.

3. $f : [0, 2] \to \mathbb{R}$ is differentiable & $f(0) = 0, f(1) = 2, f(2) = 1$. Show that there exists $c \in (0, 2)$ such that $f'(c) = 0$

Applying LMV theorem to f in [0, 1], there exists $\xi \in (0, 1)$ such that

$f(1) - f(0) = (1 - 0) f'(\xi) \Rightarrow f'(\xi) = 2$

Applying LMV theorem to f in $[1, 2]$, there exists $\eta \in (1, 2)$ such that

$f(2) - f(1) = (2 - 1) f'(\eta) \Rightarrow f'(\eta) = 1 - 2 = -1$

So $f'(\xi) f'(\eta) < 0$. By Intermediate value theorem on derivative. there exists $c \in (\xi, \eta) \subset (0, 2)$ such that $f'(c) = 0$

4. If $\phi''(x) \geq 0$ for all $x \in (a, b)$, show that

$$\phi\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{2}\{\phi(x_1) + \phi(x_2)\}$$

for every pair of points x_1, x_2 in (a, b)

Let $x_2 > x_1$ & so $x_1 < \frac{x_1 + x_2}{2} < x_2$

$$\phi(x_1) + \phi(x_2) - 2\phi\left(\frac{x_1 + x_2}{2}\right) = \left\{\phi(x_2) - \phi\left(\frac{x_1 + x_2}{2}\right)\right\} - \left\{\phi\left(\frac{x_1 + x_2}{2}\right) - \phi(x_1)\right\}$$

$= \frac{1}{2}(x_2 - x_1)\{\phi'(\xi) - \phi'(\eta)\}$ for some

$\xi \in \left(x_1, \frac{x_1 + x_2}{2}\right)$ & $\eta \in \left(\frac{x_1 + x_2}{2}, x_2\right)$ by L M V theorem ... (1)

Again by hypothesis, $\phi''(x)$ exists, so by applying L M V theorem to ϕ' in $[\xi, \eta]$, there exists $c \in (\xi, \eta)$

$\phi'(\xi) - \phi'(\eta) = (\xi - \eta)\ \phi''(c) \geq 0$ by hypothesis ...... (2)

By (1) & (2), $\phi\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{2}\{\ \phi(x_1) + \phi(x_2)\}$

Note : Converse is not true. $f(x) = |x|$ fulfils the given result but $|x|$ is not derivable at 0.

5. Let f be a function such that $f(x) > 0$ for all x & $f'(x)$ be continuous at every real x. If $f'(t) \geq \sqrt{f(t)}$ for all t, show that

$$\sqrt{f(x)} \geq \sqrt{f(1)} + \frac{1}{2}(x - 1) \text{ for all } x \geq 1$$

By hypothesis, $\phi(x) = \sqrt{f(x)}$ is derivable for all $x \geq 1$. By L M V theorem there exists $\xi \in (1, x)$ such that

$\phi(x)-\phi(1)=(x-1)\,\phi'(\xi)$

$\Rightarrow \sqrt{f(x)}-\sqrt{f(1)}=(x-1).\dfrac{f'(\xi)}{2\sqrt{f(\xi)}}\geq\dfrac{1}{2}(x-1)\left(\text{as } f'(t)\geq\sqrt{f(t)}\right)$

6. On the curve $y=x^3$, find the point at which the tangent line is parallel to the chord joining the points $A(-1,-1)$ and $B(2,8)$

Let us refer to the geometrical interpretation of L M V theorem.

By L M V theorem, there exists $\xi\in(-1,2)$ such that

$f(2)-f(-1)=(2+1)\,f'(\xi)$ for some $\xi\in(-1,2)$

(taking $f(x)=x^3$ in $[-1,2]$)

$\Rightarrow 9=3\,\xi^2.\,3\Rightarrow\xi=\pm1$. Here -1 is not interior point of $[-1,2]$ or $-1\notin(-1,2)$

So $\xi=1$ i.e (1, 1) is the point at which the tangent is paralled to AB.

7. Apply mean value theorem to find derivative of a function, assuming that the derivatives which occur are continuous.

Let $F\{f(x)\}$ be the composite function

Mean value theorem is applicable to $f(x)$ and there exists $\xi\in(x,x+h)$ or $(x+h,x)$ for which

$f(x+h)=f(x)+hf'(\xi)=u+k$ say that $u=f(x)\;\&\;k=h\,f'(\xi)$

Mean value theorem is applicable to $F(u)$ and there exists $\eta\in(u,u+k)$ or $(u+k,u)$ for which $F(u+k)=F(u)+k\,F'(\eta)$

As $h\to0$, $\xi\to x$ Also $k=h\,f'(\xi)\to0$. Further as $k\to0$, $\eta\to u$. Therefore

$$\lim_{h\to0}\frac{F\{f(x+h)\}-F\{f(x)\}}{h}=\lim_{h\to0}\frac{F(u+k)-F(u)}{h}$$

$\Rightarrow \lim\limits_{h\to0}\dfrac{k\,F'(\eta)}{h}=\lim\limits_{h\to0}\,f'(\xi)\;F'(\eta)$ exists & is

$f'(x)F'(u) = f'(x)F'\{f(x)\}$

8. If f be continuous at c and $\lim\limits_{x \to c} f'(x)$ exists finitely, then show that f' is also continuous at c.

Let $\lim\limits_{x \to c} f'(x) = l\,(\in \mathbb{R})$

Hence there exists an interval $(c, c+h], h > 0$ at every point of which f' exists & so f is continuous in $(c, c+h]$. Given that f is continuous at c. So f is continuous in $[c, c+h]$ & f' exists in $(c, c+h)$. By L M V theorem, there exists

$\xi, c < \xi < x < c+h$

such that $f(x) - f(c) = (x-c) f'(\xi)$

As $\lim\limits_{x \to c} f'(x) = l$, so

$$\lim_{x \to c+0} f'(\xi) = \lim_{\xi \to c+0} f'(\xi) = l \Rightarrow \lim_{x \to c+} \frac{f(x) - f(c)}{x - c} = Rf'(c) = l$$

Similarly, considering $[c-h, c)$ & arguing as in the previous case, $Lf'(c) = l$

so, $f'(c) = l \Rightarrow \lim\limits_{x \to c} f'(x) = f'(c)$ & therefore f' is continuous at c.

Increasing & decreasing nature of function in an interval :

Result : If $f(x)$ is continuous in $[a, b]$ and $f'(x) > 0\,(\text{or} < 0)$ in (a, b), then $f(x)$ is increasing (or decreasing) function in $[a, b]$. If $f'(x) = 0$ in $(a, b), f(x)$ is constant in the interval.

Proof : We choose x_1, x_2 so that $a \le x_1 < x_2 \le b$. Applying L M V theorem to f in $[x_1, x_2]$

there exists $\xi \in (x_1, x_2)$ such that $f(x_2) - f(x_1) = (x_2 - x_1) f'(\xi)$

So $f'(\xi) > 0 \Rightarrow f(x_2) > f(x_1)$

$(f'(\xi) < 0 \Rightarrow f(x_2) < f(x_1))$

It is true for every pair of points x_1, x_2 of $[a, b]$. So if $f'(x) > 0$ for all x. f is increasing in $[a, b]$ & if $f'(x) < 0$ for all x, f is decreasing in $[a, b]$

If $f'(x) = 0$ then $f(x_1) = f(x_2)$ & so $f(x)$ is constant in $[a, b]$

Examples : 1. If $0 < x < 1$, $2x < \log\frac{1+x}{1-x} < \frac{2x}{1-x}$

Let $f(x) = \log\frac{1+x}{1-x} - 2x, 0 \le x < 1$

$$f'(x) = \frac{1}{1+x} + \frac{1}{1-x} - 2 = \frac{2x^2}{1-x^2} > 0 \text{ for all } x \in (0,1)$$

Next let $g(t) = \log t$, $t \in [1-x, 1+x]$, $0 < x < 1$. Applying LMV theorem to $g(t)$ in $[1-x, 1+x]$, there exists $\xi \in (1-x, 1+x)$

for which $g(1+x) - g(1-x) = 2x\, g'(\xi) = \frac{2x}{\xi}$

$$\Rightarrow g(1+x) - g(1-x) = \log(1+x) - \log(1-x) < \frac{2x}{1-x}, 0 < x < 1$$

2. Show that $\frac{2}{\pi} < \frac{\sin x}{x} < 1$ when $0 < x < \frac{\pi}{2}$

Let us construct $g : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$ as follows :

$$g(x) = \begin{cases} \frac{\sin x}{x}, & 0 < x \le \frac{\pi}{2} \\ 1 & x = 0 \end{cases}$$

So g is continuous in $\left[0, \frac{\pi}{2}\right]$ & $g'(x) = \frac{x\cos x - \sin x}{x^2}$ exists in $\left(0, \frac{\pi}{2}\right)$

Let $t(x) = x\cos x - \sin x$ in $\left[0, \frac{\pi}{2}\right]$ & so

$t'(x) = -x\sin x + \cos x - \cos x < 0$ in $\left(0, \frac{\pi}{2}\right)$

So $t(x) < t(0)$ or $x\cos x - \sin x < 0$. Hence $g'(x) < 0$ in $\left(0, \frac{\pi}{2}\right)$

$\Rightarrow g\left(\frac{\pi}{2}\right) < g(x) < g(0)$ & hence $\frac{2}{\pi} < \frac{\sin x}{x} < 1$ in $\left(0, \frac{\pi}{2}\right)$

3. Show that $\frac{\tan x}{x} > \frac{x}{\sin x}, 0 < x < \frac{\pi}{2}$

Let $f(x) = \tan x \sin x - x^2, 0 \le x < \frac{\pi}{2}$

f is continuous in $[0, p]\left(p < \frac{\pi}{2}\right)$ &

$f'(x) = \sec^2 x \sin x + \sin x - 2x = t(x)$ in $[0, p]$

$t'(x) = 2\sec^2 x \tan x. \sin x + \sec^2 x. \cos x + \cos x - 2$

$= \left(\sqrt{\sec x} - \sqrt{\cos x}\right)^2 + 2\sin^2 x \sec^3 x$

$t(x)$ is continuous in $[0, p]$ & $t'(x)$ exists in $(0, p)$. Also $t'(x) > 0$ in $(0, p)$

$\Rightarrow t(x) > t(0), x > 0$ & so $f'(x) > 0 \Rightarrow f(x) > f(0), 0 < x < \frac{\pi}{2}$

Consequently, $\frac{\tan x}{x} > \frac{x}{\sin x},\ 0 < x < \frac{\pi}{2}$

4. Let $\phi(x) = f(x) + f(1-x)$ & $f''(x) < 0$ in $[0, 1]$. Show that $\phi(x)$ is monotonic increasing in $\left[0, \frac{1}{2}\right]$ and monotonic decreasing in $\left[\frac{1}{2}, 1\right]$

By hypothesis, $\phi'(x) = f'(x) - f'(1-x)$

Applying L M V theorem to f' in $[x, 1-x]$ or in $[1-x, x]$, there exists $\xi \in (x, 1-x)$ or $(1-x, x)$ such that

$f'(x) - f'(1-x) = (2x-1) f''(\xi)$

By hypothesis $f''(\xi) < 0$ & so $f'(x) - f'(1-x) \begin{cases} \geq 0 & , 0 \leq x \leq \frac{1}{2} \\ \leq 0 & , \frac{1}{2} \leq x \leq 1 \end{cases}$

So $\phi(x)$ is increasing in $\left[0, \frac{1}{2}\right]$ & is decreasing in $\left[\frac{1}{2}, 1\right]$

5. Show that $\cos x + x \sin x > 1,\ x \in \left(0, \frac{\pi}{2}\right)$

Let $f(x) = \cos x + x \sin x,\ 0 \leq x \leq \frac{\pi}{2}$

$f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$, $f'(x) = -\sin x + \sin x + x \cos x$ exists in $\left(0, \frac{\pi}{2}\right)$

Also $f'(x) > 0$ in $\left(0, \frac{\pi}{2}\right)$. So $f(x) > f(0),\ 0 < x < \frac{\pi}{2}$

$\Rightarrow \cos x + x \sin x > 1,\ 0 < x < \frac{\pi}{2}$

6. Show that $f(x) = \tan^{-1} x$ defined on $(-\infty, \infty)$ is uniformly continuous & f' is also uniformly continuous.

We note that $f'(x) = \frac{1}{1+x^2}$ exists for all $x \in \mathbb{R}$

Let us consider any pair of points x, y of $\mathbb{R}$. By L M V theorem, there exists $\xi \in (x, y)$ such that

$|f(x) - f(y)| = |x-y| \frac{1}{1+\xi^2} < |x-y| < \delta$ for any pair of points $x, y \in \mathbb{R}$ satisfying $|x-y| < \delta$, δ depends only on ε. So f is uniformly continuous on $\mathbb{R}$.

Again $|f'(x)-f'(y)|=|x-y|\,f''(\eta)$ for some $\eta\in(x, y)$ (by L M V theorem) ...(1)

$f''(x)=\dfrac{-2x}{1+x^2}$ & so $|f''(x)|<2$ for all x ...(2)

For if $|x|<1$, $\dfrac{x^2+1}{2}\geq |x|\Rightarrow |f''(x)|<2$

& if $|x|>1$, $|f''(x)|<\left|\dfrac{2x}{x^4}\right|<2$

Recalling (2), $|f'(x)-f'(y)|<2|x-y|<\varepsilon$ whenever $|x-y|<\delta$, $\delta=\dfrac{\varepsilon}{2}$

for any pair of points x, y of $\mathbb{R}$

Hence f' is uniformly continuous on $\mathbb{R}$

7. Find all possible positive solutions of $x^2+y^2=u^2+v^2$, $x^3+y^3=u^3+v^3$ where u, v be fixed positive constants.

Obiviously $x=u, y=v$ and $x=v, y=u$ are two solutions of the system.

Let $x_1=x^3$, $y_1=y^3$, $u_1=u^3$, $v_1=v^3$

We consider the function $f(t)=t^{2/3}$ in (i) $[u_1, x_1]$ (ii) $[v_1, y_1]$

By L M V theorem, there exists $t_1\in(u_1, x_1)$ & $t_2\in(v_1, y_1)$ such that

$x_1^{2/3}-u_1^{2/3}=(x_1-u_1)\dfrac{2}{3}t_1^{-1/3}$ & $v_1^{2/3}-y_1^{2/3}=(v_1-y_1)\dfrac{2}{3}t_2^{-1/3}$ ...(1)

Given $x_1+y_1=u_1+v_1$ & $x_1^{2/3}+y_1^{2/3}=u_1^{2/3}+v_1^{2/3}$ so (1) $\Rightarrow t_1=t_2$

But $t_1\in(u_1, x_1)$ & $t_2\in(v_1, y_1)$. So $t_1\neq t_2$. So $x=u$, $y=v$ & $x=v$, $y=u$ are only solutions.

Exercise :

1. Show that $0<[\log(1+x)]^{-1}-x^{-1}<1$, $x>0$

2. Show that $0 < x^{-1}\log\left(\frac{e^x - 1}{x}\right) < 1,\ x > 0$

3. Let $f:[1,3] \to \mathbb{R}$ be a continuous function that is derivable in (1, 3) with derivative $f'(x) = |f(x)|^2 + 4$ for all $x \in (1,3)$

State with reasons, whether $f(3) - f(1) = 5$ is true or false.

4. If $f''(x)$ exists in $[a,b]$ and $\frac{f(c)-f(a)}{c-a} = \frac{f(b)-f(c)}{b-c}$ for some $c \in (a,b)$, show that there exists at least one point $\xi \in (a,b)$ for which $f''(\xi) = 0$.

5. If $f'(x)$ exists for $a < x \le b$ and $|f(x)| \to \infty$ as $x \to a$, then show that $|f'(x)| \to \infty$ as $x \to a$.

(Hints Apply LMV Theorem to f in [x, b]).

6. Let f be continuous in [0,1] and differentiable in (0,1). If f' be monotonic increasing in (0,1), prove that $F(x) = \frac{f(x)}{x}$ is monotonic increasing in (0,1).

7. Show that $\tan^{-1}x_2 - \tan^{-1}x_1 < x_2 - x_1$ where $x_2 > x_1$

8. Determine the intervals of monotonicity for the following functions :

(i) $f(x) = 2x^3 - 9x^2 - 24x + 7$

(ii) $f(x) = 4x^3 - 21x^2 + 18x + 20$

(iii) $f(x) = \sin x + \cos x \text{ in } [0, 2\pi]$

9. Show that :

(a) $x - \frac{x^3}{3} < \tan^{-1}x < x - \frac{x^3}{6},\quad 0 < x \le 1$

(b) $x - \frac{x^3}{6} < \sin x < x,\ x > 0$

10. Prove that for $0 \le p \le 1$ and for any positive a and b the inequality

$(a+b)^p \le a^p + b^p$ is valid.

(Hints : Take $f(x) = 1 + x^p - (1+x)^p$, $x \ge 0$ & then $x = \frac{a}{b}$)

11. At what value (*s*) of *b*, does the function $f(x) = \sin x - bx + c$ decrease along the entire number scale ?

12. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and that $f(0) = 0;\ f(4) = 2,\ f(6) = 2$ show that

(i) there exists $x \in (0, 4)$ such that $f'(x) = \frac{1}{2}$

(ii) there exists $x \in (0, 6)$ such that $f'(x) = \frac{2}{5}$

13. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Let $f'(x) > f(x)$ for all $x \in \mathbb{R}$ & $f(x_\circ) = 0$. Show that $f(x) > 0$ for all $x > x_\circ$

$\langle$Hints : Let $g(x) = e^{-x} f(x)$ & consider the sign of $g'(x)\rangle$

14. Let $a > b > 0$. Show that $a^{1/n} - b^{1/n} < (a-b)^{1/n}$ for all $n \ge 2$

(Consider $f(x) = x^{1/n} - (x-1)^{1/n}$, $x \ge 1$; sign of $f'(x)$ & then put $x = \frac{a}{b}$)

15. Justify the following :

(a) if $x > 0$, $x > \dfrac{5\sin x}{4 + \cos x}$

(b) if $0 < x < \pi/2$, $0 < x\sin x - \frac{1}{2}\sin^2 x < \frac{1}{2}(\pi - 1)$

(c) $\sqrt{1+x} < 4 + \dfrac{x-15}{x}$, if $x > 15$

(d) $\tan^{-1} x < \frac{\pi}{4} + \frac{x-1}{2}$, if $x > 1$

(e) $p(x-1) < x^p - 1 < px^{p-1}(x-1),\ x > 1,\ p > 1$

Cauchy's Mean value theorem :

Let $f, g : [a, b] \to \mathbb{R}$ be such that (i) both are continuous in $[a, b]$ (ii) both are derivable in (a, b) (iii) $g'(x) \neq 0$ in (a, b), then there exists at least one point $c \in (a,b)$ for which

$$\frac{f(b) - f(a)}{g(b) - f(a)} = \frac{f'(c)}{g'(c)}$$

Proof : We construct the function $F(x) = f(x) + A.g(x)$ where the constant A is to be determined from the functional relation $F(a) = F(b)$.

$F(a) = F(b) \Rightarrow -A = \frac{f(b) - f(a)}{g(b) - g(a)}$. In this connection, it is to be noted that $g(b) \neq g(a)$, for if $g(b) = g(a)$, then g would satisfy all the conditons of Rolle's theorem in $[a, b]$ and so $g'(x)$ must vanish at least once in (a,b). But condition (iii) tells otherwise. So $g(b) \neq g(a)$ & $-A$ is well-defined,

Here $F'(x) = f'(x) + Ag'(x)$ exists in (a,b) by condition (ii). Also F is continuous in $[a,b]$ by hypothesis. By construction, $F(a) = F(b)$. So F satisfies all the conditions of Rolle's theorem in $[a,b]$. By Rolle's theorem, there exists at least one point $c \in (a,b)$ for which $F'(c) = 0$.

So $F'(c) = 0 \Rightarrow -A = \frac{f'(c)}{g'(c)}$. Hence $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$

Note : Putting $g(x) = x$ in $[a,b]$, we get LMV theorem.

Examples : (1) If f' exists in [0,1], show that $f(1) - f(0) = \frac{f'(x)}{2x}$ has at least one solution in (0,1).

We take $g(x)=x^2$ in [0,1]. Both *f*, *g* are continuous in [0,1], are derivable in (0,1) & $g'(x)\neq 0$ in (0,1). By Cauchy's mean value theorem, there exists at least one point $c\in(0,1)$ such that $\frac{f(1)-f(0)}{g(1)-g(0)}=\frac{f'(c)}{g'(c)}\Rightarrow f(1)-f(0)=\frac{f'(c)}{2c}$ & so c is a solution of $f(1)-f(0)=\frac{f'(x)}{2x}$.

2. Let $f:[a,b]\to\mathbb{R}$ be continuous in $[a,b]$, derivable in (a,b) where $0<a<b$. Show that for some $c\in(a,b)$

$$f(b)-f(a)=cf'(c)\log\left(\frac{b}{a}\right)$$

We take $g(x)=\log x$ in $[a,b], 0<a<b$. Applying CMV theorem to *f*, *g*, in [*a*, *b*] there exists $c\in(a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}\Rightarrow\frac{f(b)-f(a)}{\log\left(\frac{b}{a}\right)}=cf'(c)$$

3. Let *f*, *g* be differentiable on [0,2] such that $f(0)=2, f(2)=5$, $g(2)\neq 0, g(0)=0, f'(x)=g'(x)(\neq 0)$ in (0, 2). Find $g(2)$

By C M V theorem, there exists $c\in(0,2)$ such that $\frac{f(2)-f(0)}{g(2)-g(0)}=\frac{f'(c)}{g'(c)}$

$$\Rightarrow\frac{5-0}{g(2)-0}=1\Rightarrow g(2)=5$$

3.5 Taylor's Theorem

Let $f:[a,b]\to\mathbb{R}$ be such that (i) $f^{(n-1)}(x)$ is continuous in $[a,b]$ (ii) $f^{(n)}(x)$ exists in (a,b).

Then there exists $\theta \in (0,1)$ such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!} f''(a) + \ldots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n$$

where

$$R_n = \begin{cases} \dfrac{(b-a)^n (1-\theta)^{n-p}}{(n-1)!\,p} f^n\left[a+\theta(b-a)\right],\ p \in \mathbb{N}\ (\text{schlomilch – Röche'sform}) \\ \dfrac{(b-a)^n (1-\theta)^{n-1}}{(n-1)!} f^n\left[a+\theta(b-a)\right] (\text{Cauchy's form}) \\ \dfrac{(b-a)^n}{n!} \qquad f^n\left[a+\theta(b-a)\right] (\text{Lagrange's form}) \end{cases}$$

Proof : Continuity of f^{n-1} in $[a,b]$ implies the existence and continuity of $f, f', f'', \ldots f^{n-2}, f^{n-1}$ in $[a,b]$

We construct $\phi : [a,b] \to \mathbb{R}$ as follows :

$$\phi(x) = f(x) + (b-x)f'(x) + \frac{(b-x)^2}{2!} f''(x) + \ldots + \frac{(b-x)^{n-1}}{(n-1)!} f^{(n-1)}(x) + \lambda(b-x)^p$$

where λ is a constant to be determined from the functional relation $\phi(b) = \phi(a)$.

ϕ is continuous in $[a,b]$ by hypothesis (i)

$$\phi'(x) = f'(x) - f'(x) + (b-x)f''(x) - \ldots + \frac{(b-x)^{n-1}}{(n-1)!} f^n(x) - \lambda p(b-x)^{p-1}$$

exists in $(a,\ b)$

By construction, $\phi(a) = \phi(b)$. So $\phi(x)$ satisfies all the conditions of Rolle's theorem in $[a,b]$.

Therefore, by Rolle's theorem, there exists $c \in (a,\ b)$ such that $\phi'(c) = 0$

$$\Rightarrow \frac{(b-c)^{n-1}}{(n-1)!} f^n(c) = \lambda p(b-c)^{p-1} \Rightarrow \lambda = \frac{(b-c)^{n-p}}{(n-1)!\,p} f^n(c)$$

Therefore $\phi(b) = \phi(a)$ implies

$$f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2!}f''(a)+\ldots$$

$$+\frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{(b-c)^{n-p}(b-a)^p}{(n-1)!p}f^n(c)$$

As $a<c<b$ we can write $c=a+\theta(b-a)$ for some $\theta\in(0,1)$.

We write $R_n=\frac{(b-a)^n(1-\theta)^{n-p}}{(n-1)!p}f^n\left[a+\theta(b-a)\right]$ (schlomilch & Röche's form)

For $p=1$, $R_n=\frac{(b-a)^n(1-\theta)^{n-1}}{(n-1)!}f^n\left[a+\theta(b-a)\right]$ (Cauchy's form)

For $p=n$, $R_n=\frac{(b-a)^n}{n!}f^n\left[a+\theta(b-a)\right]$ (Lagrange's form)

Note : 1. The relevance of these forms by taking $p=n$ & $p=1$ will be discussed in the subsequent results.

2. The readers should note the particular forms of this theorem (also known as Generalised mean value theorem) by taking $n=2,3$ etc for solution of problems.

3. Taking $b=a+h$,

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{n!}f^n(a+\theta h)$$

for some $\theta\in(0,1)$

Problems : (i) Let $f^{n+1}(x)$ be continuous and $\neq 0$, the number θ which occurs in the Lagrange's form of remainder of Taylor's theorem, viz, $\frac{h^n}{n!}f^n(a+\theta h)$ tends to $\frac{1}{n+1}$ as $h\to o+$.

We know that—

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(x)+\frac{h^n}{n!}f^n(x+\theta h)$$ for some $\theta\in(0,1)$.

By hypothesis $f^{n+1}(x)$ exists & so

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\ldots+\frac{h^n}{n!}f^{(n)}(x)+\frac{h^{n+1}}{(n+1)!}f^{(n+1)}(x+\theta'h)$$

for some $\theta'\in(0,1)$

These two imply $f^n(x+\theta h)=f^n(x)+\frac{h}{n+1}f^{(n+1)}(x+\theta'h)$

By LMV theorem $f^n(x+\theta h)-f^n(x)=\theta hf^{(n+1)}(x+\theta\theta''h)$ for some $\theta''\in(0,1)$

So $\theta f^{(n+1)}(x+\theta\theta''h)=\frac{f^{(n+1)}(x+\theta'h)}{n+1}$

As $f^{n+1}(x)$ is continuous by hypothesis & as $f^{n+1}(x)\neq 0$ we get $\lim_{h\to 0}\theta=\frac{1}{n+1}$

2. For $x>0$, show that $0\le \sin x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\right)\le\frac{x^9}{9!}$

Let $f(x)=\sin x$. Then $f^{(n)}(x)=\sin(x+\frac{n\pi}{2})$ for all $x\in\mathbb{R},\ n\in\mathbb{N}$

These $f^{(n)}(x)$'s are continuous for all *x*.

For $x>0$ we have

$$f(x)=f(0)+x\,f'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^{n-1}}{(n-1)!}f^{n-1}(0)+\frac{x^n}{n!}f^n(\theta x)$$

for some $\theta\in(0,1)$ (taking a = 0, b = x in Taylor's theorem with Lagrange's form of remainder)

We take $n=7\ \&n=9$ respectively :

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\cos(\theta_1 x)\text{ for some }\theta_1\in(0,1)$$

$$\& \ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \cos(\theta_2 x) \text{ for some } \theta_2 \in (0,1)$$

As $-1 \le \cos(\theta_k x) \le +1$ for $k = 1, 2$ (here), so for $x > 0$, we get

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \le \sin x \le x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

Expansion of functions :

Taylor's infinite series suppose f possesses continuous derivatives of every order in $[a, a+h]$

Let $S_n = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a)$

Then $f(a+h) = S_n + R_n$

If now it is given that $R_n \to 0$ as $n \to \infty$, then $\lim_{n\to\infty} S_n = f(a+h)$

Hence under the condition that $\lim_{n\to\infty} R_n = 0$, the infinite series

$f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^n}{n!} f^n(a) + \ldots$ converges to $f(a+h)$

This result can be stated in the following way also.

Let f be defined in some open interval $I(\subset \mathbb{R})$ containing 'a' and that derivatives of every order of f exist & be throughout I. Let there exist $M \in \mathbb{R}^+$ such that $|f^n(t)| \le M$ for all $t \in I$ and for all $n \in \mathbb{N}$, then following Lagrange's form of remainder,

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + \frac{(x-a)^{n+1} f^{(n+1)}(c)}{(n+1)!} \text{ for some } c \in I.$$

As $n \to \infty$, the upper bound in RHS tends to zero. So taking $n \to \infty$

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} f^n(a)$$

Maclaurin's infinite series :

If f possess continuous derivatives of every order in $[0,h]$ and $x \in [0,h]$ and if further $\lim_{n\to\infty} R_n = 0$, then $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^n(0) + \ldots\infty$

Expansion of some elementary functions :

I. $f(x) = \sin x, x \in \mathbb{R}$

For all $n \in \mathbb{N}$, $f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right)$ & these derivatives are continuous

R_n in Lagrange's form $R_n\big|_L = \frac{x^n}{n!}f^n(\theta x) = \frac{x^n}{n!}\sin\left(\frac{n\pi}{2} + \theta x\right)$ for some $\theta \in (0,1)$

$\Rightarrow |R_n| \le \frac{x^n}{n!}$. As $\lim_{n\to\infty}\frac{x^n}{n!} = 0$, so $\lim_{n\to\infty} R_n = 0$

Hence Maclaurins expansion is valid here & so

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^n(0) + \ldots$$

$$\Rightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \ldots,\ x \in \mathbb{R}$$

II. $f(x) = e^x,\ x \in \mathbb{R}$

Here $f^{(n)}(x) = e^x$ continuous for all $x \in \mathbb{R}$

$$R_n\big|_L = \frac{x^n}{n!}f^n(\theta x) = \frac{x^n}{n!}e^{\theta x},\ \theta \in (0,1)$$

$e^{\theta x} < e^x$ and $\lim_{n\to\infty}\frac{x^n}{n!} = 0 \Rightarrow \lim_{n\to\infty} R_n = 0$

so $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^n(0) + \ldots$

$$\Rightarrow e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,\ x \in \mathbb{R}$$

III. $f(x) = \log(1+x), \quad -1 < x \le 1$

$f^{n}(x) = \dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n}, x > -1$ & these f_n's are continuous, $-1 < x \le 1$

To consider $\lim\limits_{n\to\infty} R_n$

Case I : Let $0 < x < 1$

$$R_n\big|_L = \frac{x^n}{n!} f^{(n)}(\theta x) \text{ for } \theta \in (0,1)$$

$$= \frac{(-1)^{n-1}}{n}\left(\frac{x}{1+\theta x}\right)^n \text{for some } \theta \in (0,1)$$

Here $0 < \dfrac{x}{1+\theta x} < 1$, so $\lim\limits_{n\to\infty}\left(\dfrac{x}{1+\theta x}\right)^n = 0$. Also $\lim\limits_{n\to\infty}\dfrac{1}{n} = 0$

$\Rightarrow \lim\limits_{n\to\infty} R_n = 0$

Case II $-1 < x < 0$

(In this case, it is not possible to ascertain $\lim\limits_{n\to\infty} R_n$ if R_n be considered in Lagrange's form

To substantiate this claim, let $x = \dfrac{-3}{4}$, $0 < \theta < \dfrac{1}{3}$. We see that $R_n \nrightarrow 0$ as $n \to \infty$)

We take R_n in Cauchy's form = $R_n\big|_c = \dfrac{x^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(\theta x)$ for some $\theta \in (0,1)$

Here $R_n\big|_C = (-1)^{n-1} x^n \dfrac{1}{1+\theta x}\left(\dfrac{1-\theta}{1+\theta x}\right)^{n-1}$

Also $|x| < |\Rightarrow \lim\limits_{n\to\infty} x^n = 0$. Again $\dfrac{1}{1+\theta x} < \dfrac{1}{1-|x|}$

consequently $\lim\limits_{n\to\infty} R_n = 0$

Thus the conditions for Maclaurin's expansion of $\log(1+x)$are satisfied.

Conseqently $f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+...+\frac{x^n}{n!}f^n(0)+...$

& $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-...+\frac{(-1)^{n-1}x^n}{n}+...$

If $x=1$ the series in RHS is $1-\frac{1}{2}+\frac{1}{3}...+\frac{(-1)^{n-1}}{n}+...$ which is Alternating series & is convergent by Leibnitz lest.

So the region of validity of above expansion of $\log(1+x)$ is $-1<x\leq 1$

Iv. $f(x)=(1+x)^m$ where m is any real number other than positive integer.

(If $n\in\mathbb{N}$, the series will be finite series expansion having $(n+1)$ terms)

Here $f^{(n)}(x)=m(m-1)(m-2)...(m-n+1)(1+x)^{m-n}$

We take $|x|<1$ & f^n's are continuous in $-1<x<1$.

$R_n|_C=\frac{x^n}{(n-1)!}(1-\theta)^{n-1}f^n(\theta x)$ for some $\theta\in(0,1)$.

$=\frac{m(m-1)(m-2)...(m-n+1)}{(n-1)!}x^n\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}, (1+\theta x)^{m-1}$

We know that $\lim_{n\to\infty}\frac{m(m-1)...(m-n+1)}{(n-1)!}x^n=0(\text{as }|x|<1)$

As $-1<x<1$, $0<\theta<1$, we have $0<\frac{1-\theta}{1+\theta x}<1$ & so $\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}\to 0$ as $n\to\infty$

If $(m-1)$ be positive, $0<(1+\theta x)^{m-1}<2^{m-1}$ & if $(m-1)$ be negative,

$(1+\theta x)^{m-1}<(1-|x|)^{m-1}$. As result $R_n|_C\to 0$ as $n\to\infty$

The conditions for the Maclaurin's expansion of $(1+x)^m$ are satisfied & so

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+...+\frac{x^n}{n!}f^n(0)+...$$

$$\Rightarrow (1+x)^m=1+mx+\frac{m(m-1)}{2!}x^2+...+\frac{m(m-1)...(m-n+1)}{n!}x^n...,|x|<1.$$

Particular case $\frac{1}{ax+b}=\left(1+\frac{ax}{b}\right)^{-1}.\frac{1}{b}\ (b\neq 0,\ a\neq 0)$

So the series can be deduced from above.

Application to approximate Calculations :

Examples :

1. Compute the approximate value of $\sqrt[4]{83}$ accurate to six decimal places.

We note that $\sqrt[4]{83}=\sqrt[4]{81+2}=3\left(1+\frac{2}{81}\right)^{1/4}$

By the expansion of $(1+x)^m$, taking $\frac{2}{81}$ in place of x & $\frac{1}{4}$ in place of m.

the expansion is

$$3\left[1+\frac{1}{4}\frac{2}{81}+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}\left(\frac{2}{81}\right)^2+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)}{3!}\left(\frac{2}{81}\right)^3+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)\left(\frac{1}{4}-3\right)}{4!}\left(\frac{2}{81}\right)^4+...\right]$$

$$=3\left[1+\frac{1}{162}+\frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{2}\frac{2^2}{(81)^2}+\frac{\frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)}{6}\frac{2^3}{(81)^3}+...\right]$$ & this can be computed.

2. Compute the approximate value of $\cos 5^\circ$

As in case of sinx, Maclaurin's expansion for $\cos x, x\in\mathbb{R}$, can be deduced as follows :

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$x = 5^\circ = \pi/36$ & putting $x = \pi/36$ & confining upto 2nd order terms,

$$\cos x \approx 1 - \frac{x^2}{2} \approx 1 - \left(\frac{\pi}{36}\right)^2 \text{ then}$$

$$R_4(x) = \left|\frac{\cos\theta x}{4!} x^4\right| \le \frac{x^4}{4!}\left(= \frac{1}{4!}\left(\frac{\pi}{36}\right)^4\right) \text{ etc.}$$

3.6 Summary

In this unit, we have examined the concepts of derivative, differentiability and differential. We have also studied the Rolle's theorem, Lagrange's Mean Value Theorem, Cauchy's Mean Value Theorem, Taylor's Theorem. We have further developed the Maclaurin's infinite series to expansion of some elementary functions such as e^x $\sin x$, $\log(1+x)$, $(1+x)^m$, etc. We have explained the Young's form of Taylor's Theorem.

3.7 Exercise

1. Expand $f(x) = \sin^2 x - x^2 e^{-x}$ in positive integral powers of x upto the terms of fourth order.

2. Expand $f(x) = \ln(1+\sin x)$ upto the fourth order terms.

3. Show that $\sin(\alpha + h)$ differs from $\sin\alpha + h\cos\alpha$ by not more than $\frac{h^2}{2}$

4. Expand $\ln\cos x$ upto the term containing x^4

5. If $p(x) = x^5 - 2x^4 + x^3 - x^2 + 2x - 1$, show that

$$p(x) = 3(x-1) + 3(x-1)^4 + (x-1)^5$$

Young's form of Taylor's theorem :

(Note : This form of Taylor's theorem has very important & useful application in the theory of maxima-minima...)

If a function f be such that $f^{(n)}(a)$ exists and M is defined by the equation

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+...+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{n!}M$$

$M \to f^n(a)$ as $h \to 0$

(In equivalent form, if we write the last term as $\frac{h^n}{n!}\left[f^n(a)+\varepsilon\right]$, then $\varepsilon \to 0$ as $h \to 0$)

Proof : Existence of $f^{(n)}(a)$ implies the existence of $f, f', ... f^{(n-1)}$ in

$N(a,\delta) \equiv (a-\delta, a+\delta)$ for some $\delta > 0$

Let $\varepsilon > 0$ be any number. First we take $h > 0$. We define a function ϕ as follows.

$$\phi(h)=f(a)+hf'(a)+...+\frac{h^{n-1}}{(n-1)}f^{n-1}(a)+\frac{h^n}{n!}\left[f^n(a)+\varepsilon\right]-f(a+h),\ 0<h<\delta$$

Here $\phi(0)=\phi'(0)=...=\phi^{n-1}(0)=0$ and $\phi^n(0)=\varepsilon>0$

Since $\phi^n(0)>0$ & $\phi^{n-1}(0)=0$, we see that there exists $\delta_1, 0<\delta_1<\delta$, such that $\phi^{(n-1)}(h)>0$ when $0<h<\delta_1$

Again $\phi^{n-1}(h)>0$ in $0<h<\delta_1$, $\Rightarrow \phi^{n-2}(h)>0, 0<h<\delta_1$

Proceeding in this way, we get $\phi(h)>0$ when $0<h<\delta_1$

Thus when $0<h<\delta_1$, we get

$$f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+...+\frac{h^{n-1}}{(n-1)}f^{(n-1)}(a)+\frac{h}{n!}\left[f^n(a)+\varepsilon\right]-f(a+h)>0 \quad ...(1)$$

Similarly, we can show that there exists $0<\delta_2<\delta$ such that for $0<h<\delta_2$.

$$f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{h^n}{n!}\left[f^n(a)-\varepsilon\right]-f(a+h)<0 \ldots(2)$$

Let $\eta=\min\{\delta_1,\delta_2\}$, so for $0<h<\eta$, both (1) & (2) hold

Taking into account the given relation, we get

$f^{(n)}(a)-\varepsilon<M<f^{(n)}(a)+\varepsilon$ when $0<h<\eta$

$\Rightarrow \lim_{h\to 0+} M=f^n(a)$

Taking $h<0$ arguing as before, we can get $\lim_{h\to 0} M=f^n(a)$

Combining $\lim_{h\to 0} M=f^n(a)$.

Unit-4 □ Maxima-Minima of a Function

Structure

4.0 Objectives

This unit gives

- The concept of maxima-minima of a function
- Test of maxima and minima of a function using first derivation test
- Some miscellaneous exercise will also be introduced of the end of this unit

4.1 Introduction

The maxima and minima of a function, known collectively as extrema, are the largest and smallest value of the function. In this chapter we have shown how differentiation can be used to find the extrema values of a function.

4.2 Maxima-Minima of a function

Let $f: I \to \mathbb{R}$ where I denote any interval $\subset \mathbb{R}$.

f is said to have a relative maximum (relative minimum) at $c \in I$ if there exists a neighbourhood V of c such that $f(x) \leq f(c)\left(f(x) \geq f(c)\right)$ for all x in $V \cap I$. If f has either relative maximum or relative minimum at c, we say that f has a relative extremum at c.

Interior extremum Theorem :

Let c be an interior point of interval I at which $f : I \to \mathbb{R}$ has a relative extremum. If the derivative of f exists at c. then $f'(c) = 0$

If possible, let $f'(c) > 0$. Then there exists a neighbouhood $V(\subset I)$ of c such that

$$\frac{f(x) - f(c)}{x - c} > 0(x \in V, x \neq c)$$

So if $x \in V$, $x > c$, $f(x) - f(c) > 0$ in V and if for $x \in V$, $x < c$

$f(x) - f(c) < 0$. As a result $f(x) - f(c)$ does not maintain the same sign througout the both-sided neighbourhood of c. As a result f has no extremum at c. Thus we arrive at a contradiction. So $f'(c) \not> 0$. As a result, $f'(c) = 0$

Note : f may not be derivable at an extremum. For example $f(x) = |x|$ has minimum at $x = 0$ but f' does not exist at $x = 0$

2. At a point of domain of f, $f'(x) = 0$ does, not ensure the existence of extrenum at that point.

For example, $f(x) = x^{2n+1}, n \in \mathbb{N}$

Note that $f(x) = 0$ at $x = 0$. But $f(x) - f(0) > 0$ if $x > 0$, $f(x) - f(0) < 0$ if $x < 0$ i.e $f(x) - f(0)$ does not maintain the same sign in both sided neighbourhood of 0.

Sufficient condition for maximum/minimum of function.

Let c be an interior point of the domain I of f

Let (i) $f^{(n)}(c)$ exist and $f^{(n)}(c) \neq 0$

(ii) $f'(c) = f''(c) = ... = f^{(n-1)}(c) = 0$

Then if n is odd, f has no extremum at c.

But if n be even, f has an extremum at c and $f(c)$ is maximum or minimum at c according as $f^{(n)}(c) < 0$ or $f^{(n)}(c) > 0$

Proof : Recalling Young's form of Taylor's theorem,

$$f(c+h)-f(c)=hf'(c)+\frac{h^2}{2!}f''(c)+...+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(c)+\frac{h^n}{n!}M \Rightarrow$$

$$M \to f^{(n)}(c) \text{ as } h \to 0$$

By (ii) $f(c+h)-f(c)=\frac{h^n}{n!}M$ ....(1)

Since $M \to f^{(n)}(c)$ as $h \to 0$, there exists $\delta > 0$ such that for $0<|h|<\delta$, M and $f^{(n)}(c)$ have the same sign.

So (1) $\Rightarrow$ when n is even, $f(c+h)-f(c)$ and M have the same sign. If $f^{(n)}(c)>0$, M & hence $f(c+h)-f(c)>0$ which implies that $f(c)$ is minimum. If $f^{n}(c)<0$, $f(c+h)-f(c)<0$, meaning thereby that $f(c)$ is maximum. If n be odd, $M<0$ & as a result, $f(c+h)-f(c)$ changes sign with the change in the sign of h. So if n be odd, $f(c)$ is not an extreme value.

4.3 First derivative test

Let $f:[a,b] \to \mathbb{R}$ be continuous in $[a,b]$. Let $a<c<b$ and let f be differentiable in both (a,c) and (c,b). Then

(i) if there exists $\delta > 0$ such that $f'(x) \geq 0$ in $(c-\delta, c)$ and $f'(x) \leq 0$ in $(c, c+\delta)$, then f has a local maximum at c.

(ii) if there exists $\delta > 0$ such that $f'(x) \leq 0$ in $(c-\delta, c)$ and $f'(x) \geq 0$ in $(c, c+\delta)$, then f has a local minimum at c.

(iii) if $f'(x)$ maintains the same sign in both $(c-\delta, c)$ & $(c, c+\delta)$, then f has no extremum at c.

Proof : By hypothesis, f satisfies the conditions of L M V theorem in both $[c-\delta, c]$ & in $[c, c+\delta]$. So by L M V theorem, there exists $\xi \in (x,c)(\subset (c-\delta, c))$ and $\eta \in (c,x)(\subset (c, c+\delta))$ for which

$f(c)-f(x)=(c-x)f'(\xi)$ & $f(x)-f(c)=(x-c)f'(\eta)$

(i) given that $f'(x)\geq 0$ in $(c-\delta,c)$ & so $f(c)-f(x)\geq 0$

and $f'(x)\leq 0$ in $(c,c+\delta)\Rightarrow f(x)-f(c)\leq 0$

so in both cases, $f(c)\geq f(x)$ in $N(c,\delta)\cap[a,b]$

$\Rightarrow f$ has a local maximum at c.

(ii) given that $f'(x)\leq 0$ in $(c-\delta, c)$ & so $f(c)-f(x)\leq 0$

& $f'(x)\geq 0$ in $(c,c+\delta)\Rightarrow f(x)-f(c)\geq 0$ n $(c,c+\delta)$

In both cases, $f(x)-f(c)\geq 0$ or $f(x)\geq f(c)$ in $N(c,\delta)\cap[a,b]$

$\Rightarrow f$ has a local minima at c.

(iii) if $f'(x)$keeps same sign in both $(c-\delta,c)$ & in $(c,c+\delta)$, $f(x)-f(c)$ does not maintain the same sign & meaning thereby that f has no extremum at c.

Note : The conditions are sufficient but not necessary for the existence of extremum.

Let $f(x)=\begin{cases} 2x^2+x^2\sin\dfrac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$

Here $x^2\leq f(x)\leq 3x^2$ & so f has a strict local minimum at $x=0$ but $f'(x)=4x+2x\sin\dfrac{1}{x}-\cos\dfrac{1}{x}$ is not of constant sign in any deleted neighbourhood of $x=0$.

Problems on Maxima-Minima :

1. Let $f(x)=1-\sqrt{(x^2)}$ where the square root is to be taken positive. Test for the existence of maximum/minimum

Here $f(x)=\begin{cases}1-x, & if\ x\geq 0\\ 1+x, & if\ x<0\end{cases}$

In $0<x<0+\delta$, $f(x)-f(0)=1-x-1=-x<0$ & in

$0-\delta<x<0,\ f(x)-f(0)=x<0$

So in any case, $f(x)-f(0)<0$ meaning f has a maximum at $x=0$

Note : $\lim_{x\to 0+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0+}\frac{1-x-1}{x}=-1<0\Rightarrow R\,f'(0)<0$

& $\lim_{x\to 0-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0-}\frac{1+x-1}{x}=1>0\ L.f'(0)>0$

Hence f' does not exist at $x=0$)

2. P is any point on the curve $y=f(x)$ & C is a fixed point not on the curve. If the length PC is either maximum or minimum, show that the line PC is perpendicular to the tangent at P.

Let $P(x_1,y_1)$ be any point on curve $y=f(x)$ & fixed point C be (α,β).

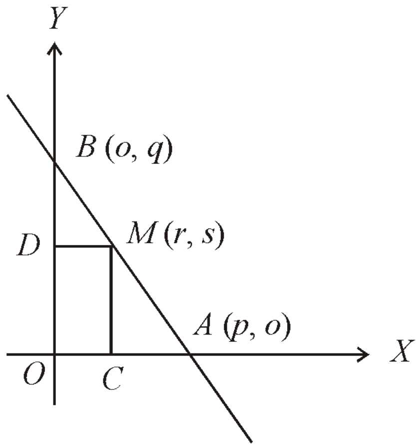
So $PC=\sqrt{\left[(x_1-\alpha)^2+(y_1-\beta)^2\right]}=\sqrt{\left[(x_1-\alpha)^2+(f(x_1)-\beta)^2\right]}$

$$\frac{d(PC)}{dx_1}=0\Rightarrow 2(x_1-\alpha)+2\left[f(x_1)-\beta\right]f'(x_1)=0\Rightarrow f'(x_1)=-\frac{x_1-\alpha}{f(x_1)-\beta}=m_1,$$

the slope of tangent at P.

Slope of $PC=m_2=\frac{f(x_1)-\beta}{x_1-\alpha}$ and hence $m_1m_2=-1$. Therefore for extremum of PC, PC is perpendicular to the tangent at P.

3. A rectangle is inscribed in a right-angled triangle so as to have one angle coincident with the right angle. Prove that its area is maximum when the opposite corner bisects the hypotenuse.



We take x-axis and y-axis as along the base & the perpendicular line of the given triangle, OCMD is the rectangle.

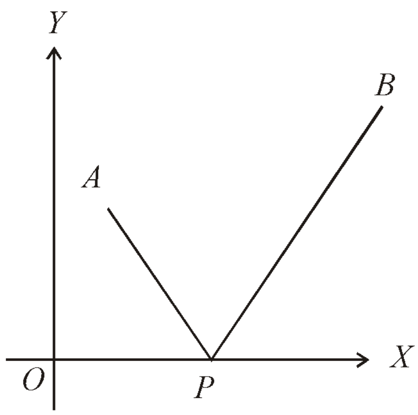
AB is $\frac{x}{p}+\frac{y}{q}=1$ & $M(r,s)$ is on $AB \Rightarrow \frac{r}{p}+\frac{s}{q}=1$

Area of rectangle = $rs = r\left(1-\frac{r}{p}\right)q = f(r)$

$f'(r)=q\left(1-\frac{2r}{p}\right)=0 \Rightarrow r = p/2$ so $f''(r)=\frac{-2q}{p}<0$

$f(r)$ is maximum when $r=\frac{p}{2},\ s=\frac{q}{2}$. Hence M is midpoint of AB.

4. Find a point on a given straight line such that the sum of its distances from two given points on the same side of the line is a minimum.



$A(a_1, b_1)$ and $B(a_2, b_2)$ are the given points. With reference to the given line as x-axis & line perpendicular to it as y-axis.

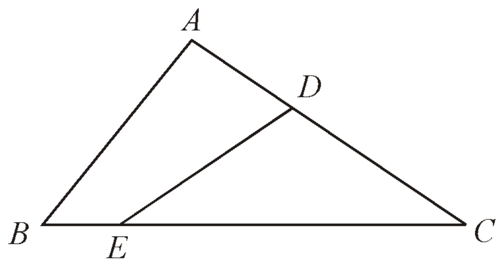
Let $f(x) = \sqrt{\left[(x-a_1)^2 + b_1^2\right]} + \sqrt{\left[(a_2 - x)^2 + b_2^2\right]}$

$f'(x) = 0 \Rightarrow x = \dfrac{a_1 b_2 + a_2 b_1}{b_1 + b_2}$

Note that $f''\left(\dfrac{a_1 b_2 + a_2 b_1}{b_1 + b_2}\right) > 0 \Rightarrow f(x)$ is minimum when $x = \dfrac{a_1 b_2 + a_2 b_1}{b_1 + b_2}$

Consequently we conclude that when $f(x)$ is minimum, the x-co-ordinate of the point on the fixed line is same as the x-co-ordinate of the point which divides AB internally in the ratio $b_1 : b_2$.

5. A person wishes to divide a triangular field by a straight fence into two equal parts. Show how it is to be done so that the fence may be of minimum length.



Let the fence be DE.

Length $|DC| = y$, length $|EC| = x$

Length $|DE| = z$

So $z^2 = x^2 + y^2 - 2xy\cos C$ ...(1)

By hypothesis, $\dfrac{1}{2}xy\sin C = \dfrac{1}{2}\cdot\dfrac{1}{2}ab\sin C \Rightarrow y = \dfrac{ab}{2x}$ ...(2)

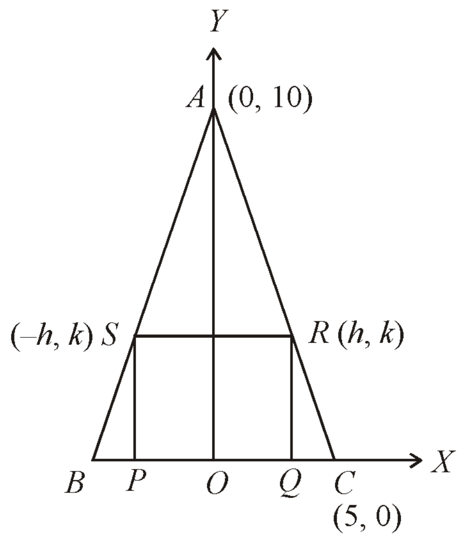
So $z^2 = x^2 + \dfrac{a^2b^2}{4x^2} - ab\cos C = f(x)$ (say)

$$f'(x) = 2x - \frac{a^2b^2}{2x^3} \Rightarrow x = \sqrt{\frac{ab}{2}}$$

$$f''(x) = 2 + \frac{3a^2b^2}{4} \Rightarrow f''\left(\sqrt{\frac{ab}{2}}\right) > 0 \Rightarrow \text{ minimum for } x = \sqrt{\frac{ab}{2}}$$

So z is minimum for $x = y = \sqrt{\frac{ab}{2}}$

6. Find the dimensions of the largest rectangle which can be inscribed in an isosceles triangle of base 10 cm & altitude 10 cm.



As $\Delta\, ABC$ is isosceles $(AB = AC)$, so median from A on BC is perpendicular on BC. We take mid point O of BC. as origin, positive side of x-axis along OC & positive side of y-axis along OA. Refering to the figure, area of rectangle A = 2hk (unit). AC is $\frac{x}{5} + \frac{y}{10} = 1$ & so $2h + k = 10 \Rightarrow k = 10 - 2h$

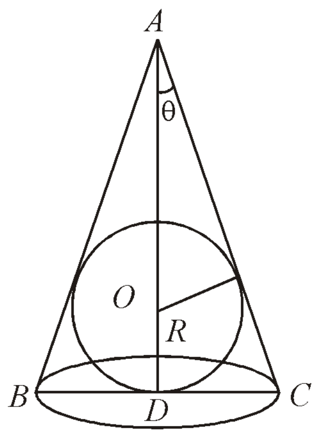
As $A = 2h(10 - 2h) = f(h)$, $f'(h) = 20 - 4.2h$

$$f'(h) = 0 \Rightarrow h = \frac{20}{8} = \frac{5}{2} \text{ (unit) \& } k = 5 \text{ (unit)}$$

$f''(h) = -8 < 0$. So A is maximum for $h = \frac{5}{2}$, $k = 5$

These give the dimensions of the largest rectangle.

7. A cone is circumscribed about a sphere of radius R. Show that when the volume of the cone is minimum, its altitude is $4R$ and its semivertical angle is $\sin^{-1}\left(\frac{1}{3}\right)$.



Let the radius of the base be x (unit) and the height of the cone be z (unit). By property of elementary geometry,

A, O, D are collinear, $BD = DC$ & $AB = AC$

From the figure $\sin\theta = \frac{R}{Z-R} = \frac{x}{\sqrt{(x^2+z^2)}} \Rightarrow x^2 = \frac{R^2 z}{z-2R}$

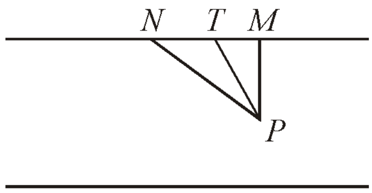
Volume $V = \frac{1}{3}\pi\, x^2 z = \frac{\pi R^2}{3} . \frac{z^2}{z-2R} = f(z)$

For extremum $f'(z) = 0$ and $\Rightarrow z = 4R$ and $f''(4R) > 0$

So V is minimum for $z = 4R$ and so $\sin\theta = \frac{1}{3}$ i.e. $\theta = \sin^{-1}\frac{1}{3}$.

8. A man in a boat $\frac{\sqrt{3}}{2}$ miles from the bank wishes to reach a village that is $5\frac{1}{2}$ miles distant along the bank from the point nearest to him. He can walk 4 m.p.h. & row 2 m.p.h. Where should he land in order to reach the village in the least time ?

Find also the time.



Let P be the position of the man & let he land at T. Let $MT = x$ miles.

$$PT = \frac{\sqrt{(4x^2+3)}}{2} \ \& \ NT = \frac{11}{2} - x$$

Let t be the total time to reach N then

$$t = \frac{\sqrt{(4x^2+3)}}{4} + \frac{11}{8} - \frac{x}{4} = f(x) \ \text{(say)}$$

For extremum $f'(x) = 0 \Rightarrow x = \frac{1}{2}$

Here $f''\left(\frac{1}{2}\right) > 0$ & so t is least for $x = \frac{1}{2}$. Then $t = 1\frac{3}{4}$ hours.

9. Prove that a conical tent of a given capacity will require the least amount of canvas when the height is $\sqrt{2}$ times the radius of the base.

Let the cone be of semi-vertical angle α & radius of its base be r (unit). Then volume $V = \frac{1}{3}\pi r^3 \cot\alpha$ & surface area $S = \pi r^2 \operatorname{cosec}\alpha$

$$\frac{dV}{d\alpha} = 0 \text{ gives } \frac{1}{3}\pi\left[3r^2 \cot\alpha \frac{dr}{d\alpha} - r^3 \operatorname{cosec}^2\alpha\right] = 0$$

$$\Rightarrow \frac{dr}{d\alpha} = \frac{r^3 \operatorname{cosec}^2\alpha}{3r^2 \cot\alpha} = \frac{r \operatorname{cosec}^2\alpha}{3\cot\alpha}$$

Also $\dfrac{dS}{d\alpha} = \pi\left[2r\ \mathrm{cosec}\alpha \dfrac{dr}{d\alpha} - r^2 \mathrm{cosec}\ \alpha \cot\alpha\right]$

putting the expression for $\dfrac{dr}{d\alpha}$, we get $\dfrac{dS}{d\alpha}$

$$= \pi\left[2r\ \mathrm{cosec}\,\alpha . \frac{r\ \mathrm{cosec}^2\alpha}{3\cot\alpha} - r^2\ \mathrm{cosec}\,\alpha \cot\alpha\right]$$

$$\frac{dS}{d\alpha} = 0 \Rightarrow \frac{2r^2\ \mathrm{cosec}^3\alpha}{3\cot\alpha} - r^2\ \mathrm{cosec}\alpha\ \cot\alpha \Rightarrow \cot\alpha = \sqrt{2} \text{ or } \alpha = \cot^{-1}\sqrt{2}$$

As α passes through the value $\cot^{-1}\sqrt{2}, \dfrac{dS}{d\alpha}$ changes its sign from negative to positive & by first derivative test, S is minimum for $\alpha = \cot^{-1}\sqrt{2}$.

Then height = $r\cot\alpha = r\sqrt{2} = \sqrt{2} \times$ radius of the base.

4.4 Exercise-I

1. Prove that the greatest acute angle at which the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ can be cut by a concentric circle is $\tan^{-1}\left[\dfrac{a^2 - b^2}{2ab}\right]$

2. Show that the maximum & minimum values of $r^2 = x^2 + y^2$ where $ax^2 + 2hxy + by^2 = 1$ are given by the quadratic $\left(a - \dfrac{1}{r^2}\right)\left(b - \dfrac{1}{r^2}\right) = h^2$

[Hints : $x = r\cos\theta,\ y = r\sin\theta \Rightarrow \dfrac{1}{r^2} = a\cos^2\theta + h\sin 2\theta + b\sin^2\theta$

$$\frac{dr}{d\theta} = 0 \Rightarrow \frac{\sin 2\theta}{2h} = \frac{\cos 2\theta}{a - b} = \frac{1}{\sqrt{\left[(a-b)^2 + 4h^2\right]}} = \frac{1}{k} \text{ (say)]}$$

3. Show that the maximum value of $\dfrac{\log x}{x}$ in $0 < x < \infty$ is $\dfrac{1}{e}$.

4. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius 'a' is $2a/\sqrt{3}$.

5. If $f(x)=(x-a)^{2n}(x-b)^{2m+1}(m,n\in\mathbb{N})$, test for the existence of extremum.

6. Find the altitude of the cone of maximum volume that can be inscribed in a sphere of radius 'a'.

7. A rectangle is drawn inside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ having sides parallel to axes of the ellipse. Show that when the rectangle is greatest, the diagonals of the rectangle will be along the conjugate diameters of the ellipse.

8. Of all triangles with the given base $2a$ unit & given area ah square unit, find that with the least perimeter.

9. At which point on the ellipse $\frac{x^2}{8}+\frac{y^2}{18}=1$ must a tangent be drawn such that the area of the triangle formed by the tangent & the co-ordinate axes is the smallest ?

10. Investigate for extremum :

(i) $f(x)=\begin{cases} -2x, & x<0 \\ 3x+5, & x\geq 0 \end{cases}$

(ii) $f(x)=\begin{cases} 2x^2+3, & x\neq 0 \\ 4, & x=0 \end{cases}$

4.5 Appendix

On monotonic functions :

In chapter II, we have just stated an important result on the continuity/discontinuity of monotone functions without giving the proof or any other property. Here we are going to discuss some properties of monotone functions :

Theorem : Let $f:[a,b]\to\mathbb{R}$ be monotonic increasing in $[a,b]$ & $a<x_0<b$. Then $\lim\limits_{x\to x_0^-} f(x)$ (or $f(x_0-)$) and $\lim\limits_{x\to x_0+} f(x)$ (or $f(x_0+)$) both exist and

$$f(x_0-)\leq f(x_0)\leq f(x_0+)$$

Proof : Let $A=\{f(x)|a<x<x_0\}$. Since f is increasing function, the set A is bounded above by $f(x_0)$. By completeness axiom of $\mathbb{R}$, the set A has the least upper bound (Sup) C (say). Then $C \le f(x_0)$.

We propose to show that $f(x_0-)$ exists & that it equals C.

Let $\varepsilon>0$ be given. As $\sup A = C$, so corresponding to ε, there exists $\delta>0$ such that $a<x_0-\delta<x_0$ & $C-\varepsilon<f(x_0-\delta)\le C$.

Since f is monotonic increasing function, we have

$x_0-\delta<x<x_0 \Rightarrow f(x_0-\delta)\le f(x)\le C$.

So $C-\varepsilon<f(x)\le C$ for $x_0-\delta<x<x_0$

$\Rightarrow f(x_0-)=C\le f(x_0)$

Next let $B=\{f(x)|x_0<x<b|\}$, since f is increasing function, the set B is bounded below by $f(x_0)$. So the set B has the greatest lower bound (inf f) d (say). Then $f(x_0)\le d$.

We propose to show that $f(x_0+)$ exists & $f(x_0+)=d$

Let $\varepsilon>0$ be given. As $d=\inf B$, corresponding to ε, there exists $\delta>0$ such that $x_0<x_0+\delta<b \Rightarrow d\le f(x_0+\delta)<d+\varepsilon$

Since f is monotonic increasing function, we have

$x_0<x<x_0+\delta \Rightarrow d\le f(x)\le f(x_0+\delta)\Rightarrow d\le f(x)<d+\varepsilon,\ x_0<x<x_0+\delta$

$\Rightarrow f(x_0+)$ exists and $f(x_0+)=d\ge f(x_0)$

consequently $f(x_0-)\le f(x_0)\le f(x_0+)$.

Remark : At the end points, $f(a)\le f(a+), f(b-)\le f(b)$

Note : (1) Let $a<x<y<b$

Then $f(x+)=\inf\limits_{x<t<b} f(t)\le \inf\limits_{x<t<y} f(t)$

$f(y-)=\sup\limits_{a<t<y} f(t)\ge \sup\limits_{x<t<y} f(t)$

As $\inf_{x<t<y} f(t) \leq \sup_{x<t<y} f(t)$, so $f(x+) \leq f(y-)$

Also if $x_1, x_2 ..., x_n$ be n interior points of $(a,b), a < x_1 < x_2 < ... < x_n < b$, we have $\sum_{k=1}^{n} [f(x_k+) - f(x_k-)] \leq f(b-) - f(a+)$

Note 2. Monotonic functions can have only discontinuity of first kind or in other words, monotonic functions can have no discontinuity of second kind.

Theorem : If $f : [a,b] \to \mathbb{R}$ be a monotonic, discontinuous function, the set E of points of discontinuity of f is atmost enumerable.

Proof : With every point x of E, we associate a rational number $r(x)$ such that $f(x-) < r(x) < f(x+)$

since $x_1 < x_2 \Rightarrow f(x_1+) \leq f(x_2-)$ we see that $r(x_1) \neq r(x_2)$ if $x_1 \neq x_2$. We have thus established a one-one correspondence between the set E & a subset of the set of rational numbers. The set of rational numbers is atmost enumerable. Hence the result follows.

Note : 3. Jump of f at a point :

We know that the jump of function f at a point c is defined by

$j_f(c) = f(c+0) - f(c-0)$

Let $f : [a,b] \to \mathbb{R}$ be increasing function. Let us now consider jumps of f at distinct points.

Let $a < p < x < q < b$. f being increasing,

$f(a) \leq f(p-0) \leq f(p+0) \leq f(x) \leq f(q-0) \leq f(q+0) \leq f(b)$

$\Rightarrow j_f(p) + j_f(q) \leq f(b) - f(a)$

$\Rightarrow$ for distinct points $p_1, p_2 ..., p_n$ in (a,b) we have

$j_f(p_1) + ... j_f(p_n) \leq f(b) - f(a)$

Hence if there are k distinct points where the jump of f is at least t, then

$k \leq [f(b) - f(a)]/t$

2. On second mean value theorem of differential calculus

Let $f:[a,b]\to\mathbb{R}$ be such that

(i) f, f' are continuous in $[a,b]$

(ii) f'' exists in (a,b)

Then there exists at least one point $c\in(a,b)$ such that

$$f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2}f''(c)$$

Proof : Let us construct $F:[a,b]\to\mathbb{R}$ by

$F(x)=f(x)+(b-x)f'(x)+A(b-x)^2$ where the constant A is to be determined from $F(b)=F(a)\Rightarrow f(b)=f(a)+(b-a)f'(a)+A(b-a)^2$ ...(1)

Continuity of $f, f', (b-x)^2 \Rightarrow$ continuity of F in $[a,b]$

$F'(x)=f'(x)-f'(x)+(b-x)f''(x)-A.2(b-x)$ exists in (a,b) by hyp...(2)

By construction $F(b)=F(a)$. So F satisfies all the conditions of Rolle's theorem in $[a,b]$

Therefore, by Rolle's theorem, there exists $c\in(a,b)$ for which $F'(c)=0$

$\Rightarrow (b-c)f''(c)-2A(b-c)=0$

$\Rightarrow A=\frac{1}{2}f''(c)$

Putting in (1) $f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2}f''(c)$

Note this result is in fact Taylor's theorem for n = 2.

3. On convex function

Definition : Let $f:I\to\mathbb{R}$ where I is some open interval $\subset\mathbb{R}$. If for pair of points $x_1, x_2\in I$ and any number $\alpha_1,\alpha_2(\geq 0), \alpha_1+\alpha_2=1$, the inequality

$f(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2)$...(1) holds, then f is said to be a convex function or convex downward. If only < holds in (1), f is stricly convex on I. If the opposite equality holds $f(\alpha_1 x_1 + \alpha_2 x_2) \geq \alpha_1 f(x_1) + \alpha_2 f(x_2)$ for any pair x_1, x_2 as stated above, f is concave or convex upward on I.

Remarks : Taking $x = \alpha_1 x_1 + \alpha_2 x_2$ when $\alpha_1 + \alpha_2 = 1$ we have

$$\alpha_1 = \frac{x_2 - x}{x_2 - x_1},\ \alpha_2 = \frac{x - x_1}{x_2 - x_1}$$

& hence $\dfrac{f(x) - f(x_1)}{x - x_1} \leq \dfrac{f(x_2) - f(x)}{x_2 - x}$ for $x_1 < x < x_2$...(2)

Theorem : A necessary & sufficient condition for $f : I \to \mathbb{R}$ that is derivable on I to be convex (downward) on I is that its derivative f' to be increasing on I. (A strictly increasing f' corresponds to strictly convex function)

Proof : Let the convex function $f : I \to \mathbb{R}$ be differentiable on I.

In (2) taking x tends first to x_1 & then to x_2, we have

$$f'(x) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq f'(x_2)$$

Applying L M V theorem to f in $[x_1, x_2]$, there exists $\xi \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(\xi)$$

So $f'(x_1) \leq f'(\xi) \leq f'(x_2)$ & so the derivative of f is monotonic.

For a strictly convex function. $f'(x_1) < f'(\xi) < f'(x_2)$ & f' is strictly monotonic.

Converse : L M V theorem $\Rightarrow f(x) - f(x_1) = (x - x_1) f'(\xi_1)$ for some $\xi_1 \in (x_1, x)$

& $f(x_2) - f(x) = (x_2 - x) f'(\xi_2)$ for some $\xi_2 \in (x, x_2)$

If $f'(\xi_1) \leq f'(\xi_2)$ then (2) follows & f is convex function.

Note : Let $f : I \to \mathbb{R}$ be twice differentiable on the open interval $I(\subset \mathbb{R})$

Then f is convex on I if $f''(x) > 0$ throughout I.

Theorem : Let f be convex on the open interval $I(\subset \mathbb{R})$. Then

(i) the limits $\lim\limits_{h\to 0-} \dfrac{f(x+h)-f(x)}{h}$ & $\lim\limits_{h\to 0+} \dfrac{f(x+h)-f(x)}{h}$ both exist for each $x \in I$

(ii) f is continuous on I.

Proof : As f is convex on I, by (2) for $x_1 < x_2 < x_3 (\text{all} \in I)$

$$\frac{f(x_2)-f(x_1)}{x_2 - x_1} \le \frac{f(x_3)-f(x_2)}{x_3 - x_2}$$

we take $x_1 = x,\ x_2 = x + h_1,\ x_3 = x + h_2$ where $0 < h_1 < h_2$

Then $\dfrac{f(x+h_1)-f(x)}{h_1} \le \dfrac{f(x+h_2)-f(x)}{h_2}$

so if $F(h) = \dfrac{f(x+h)-f(x)}{h}$, $h > 0$, then $F(h)$ increases in some interval $(0, \delta)$

so $\lim\limits_{h\to 0+} F(h)$ exists. Similarly $\lim\limits_{h\to 0-} F(h)$ exists

$$\lim_{h\to 0+} \{f(x+h)-f(x)\} = \lim_{h\to 0+} \left\{\frac{f(x+h)-f(x)}{h} h\right\} = 0$$

Similarly $\lim\limits_{h\to 0+} \{f(x+h)-f(x)\} = 0$. Hence f is continuous function on I.

Notes : The result may fail if I be not open.

$$f(x) = \begin{cases} x^2, & 0 \le x < 1 \\ 3, & x = 1 \end{cases}$$

Examples :

(1) Let $f(x) = x^{\alpha}, x > 0, \alpha \in \mathbb{R}$

$$f''(x) = \alpha(\alpha - 1)x^{\alpha-2}\begin{cases} > 0, & \text{for } \alpha < 0 \text{ or } \alpha > 1 \\ < 0, & \text{for } 0 < \alpha < 1 \end{cases}$$

So if $\alpha < 0$ or $\alpha > 1$ f is strictly convex & for $0 < \alpha < 1$, f is concave function

(2) $\sin x$ is strictly convex when $2k\pi < x < (2k+1)\pi$ & concave when $(2k-1)\pi < x < 2k\pi$.

(3) $a^x (a > 0, a \neq 1)$ is convex for $0 < a < 1,\ a > 1$

4. On Periodic function :

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be periodic on $\mathbb{R}$ if there exists a number p such that $f(x+p) = f(x)$ for all x. The least positive value of p for which $f(x+p) = f(x)$ is known as the period of f or the primitive period of f. For example, sinx, cosx are periodic functions of period 2π.

Result : (i) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with period 1. Then (i) f is bounded above & below and achieves its maximum & minimum values. (ii) there exists a real number $x_\circ$ such that $f(x_\circ + \pi) = f(x_\circ)$.

Proof : Let f_1 be the restriction of f to $[0,2]$. As $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. the ranges of f & f_1 are same. f is bounded & attains its maxima and minima there in. As f is continuous & periodic on $\mathbb{R}$. So f is bounded above & below and achieves its maximum and minimum. Let f attain its maximum & minimum at p and q respectively.

Hence $f(p+\pi) - f(p) \leq 0$ & $f(q+\pi) - f(q) \geq 0$

If the equality holds in the first case, p is desired $x_\circ$

If the equality holds in the second case, q is desired x_0.

Otherwise : Let $g(x) = f(x+\pi) - f(x), x \in \mathbb{R}$

So g is continuous & $g(p)g(q) < 0$. Applying Bolzano's theorem on continuous function to g on $[p,q]$(or on $[q,p]$), there exists $x_\circ \in \mathbb{R}$ for which $g(x_\circ)=0$ i.e $f(x_\circ+\pi)=f(x_\circ)$.

Result 2. Let $f:\mathbb{R}\to\mathbb{R}$ be continuous and periodic & let $T(>0)$ be the period. Then f is uniformly continuous on $\mathbb{R}$

Proof : Continuity of f in $[-T, 2T]$ implies f is uniformly continuous in $[-T, 2T]$. For arbitrary $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ such that for any pair of points $x,y\in[-T, 2T]$ satisfying $|x-y|<\delta$, we have $|f(x)-f(y)|<\varepsilon$ (1)

We take $0<\delta<T$

Let $x,y,\in\mathbb{R}$ satisfying $|x-y|<\delta$

There exists $n\in\mathbb{Z}$ such that $nT\le x<(n+1)T$ & so $x-nT\in[0,T]$ & $y-nT\in[-T,2T]$.

Note that $|(x-nT)-(y-nT)|=|x-y|<\delta\Rightarrow|f(x)-f(y)|<\varepsilon$

$\Rightarrow$Uniform Continuity of f on $\mathbb{R}$

Result 3. Let $f:\mathbb{R}\to\mathbb{R}$ be a periodic function. Show that if $\lim_{x\to\infty} f(x)$ exists, then f is a constant function.

Proof : Let $\lim_{x\to\infty} f(x)=l(\in\mathbb{R})$ and $T(>0)$ be the period of f. We propose to show that $f(x)=l$ for all x.

If not and if possible, let there exist $a\in\mathbb{R}$ such that $f(a)\ne l$.

Let $0<\varepsilon<\dfrac{|f(a)-l|}{10}$. As $\lim_{x\to\infty} f(x)=l$, so corresponding to above ε, there exists $G\in\mathbb{R}$ such that

$|f(x)-l|<\varepsilon$ whenever $x>G$ ...(1)

By Archemedean property of real numbers, there exists $n\in\mathbb{N}$ such that $nT>G-a$ so $nT+a>G$ ...(2)

By (1) and (2) $|f(a+nT)-l|<\varepsilon \Rightarrow |f(a)-l|<\varepsilon$ (as T is the period of f)

$\Rightarrow 10\varepsilon<\varepsilon$ but this is absurd as $\varepsilon>0$

So $f(x)=l$ for all x & as a result f is constant function

Example : $\lim\limits_{x\to\infty}\sin x$ does not exist

4.6 Summary

In this unit we have defined the term extrema of a function and shown how the differentiation can be used to find the maxima & minima. We have also studied the first derivative test for extrema and formulated a sufficient condition for extrema of a function.

4.7 Miscellanous Exercise

1. Let $f,\ g: S\to\mathbb{R}\ (S\subset\mathbb{R})$ and p be an accumulation point of S.

Let $\lim\limits_{x\to p} f(x)=l(\in\mathbb{R})$ and $\lim\limits_{x\to p} g(x)=m(\in\mathbb{R})$

Test for the existence of

(i) $\lim\limits_{x\to p}\max\{f,g\}$ (ii) $\lim\limits_{x\to p}\min\{f,g\}$

2. Using the results (i) $\left\{\left(1+\frac{1}{n}\right)^n\right\}_n$ converges to e and (ii) for $x>1$ there exists $n\in\mathbb{N}$ such that $n\le x<n+1$, show that $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$

3. Let $f:[a,\infty)\to\mathbb{R}$. Then show that $\lim\limits_{x\to\infty} f(x)$ exists if and only if for every $\varepsilon>0$, there exists $X>a$ such that

$|f(x)-f(y)|<\varepsilon$ for all $x,y>X$

4. Let $n\in\mathbb{N}$ and $\lambda>0$. Show that there exists unique $y>0$ such that $y^n=\lambda$

5. Let $f(x)=\begin{cases} x^2-2x, \text{ when } x \text{ is rational} \\ 3x-6, \text{ when } x \text{ is irrational} \end{cases}$

If $a \in \mathbb{R}$, examine whether $\lim_{x\to a} f(x)$ exists.

6. Prove or disprove : If $f(x)=\begin{cases} x, & \text{when } x \text{ is rational} \\ 1-x, & \text{when } x \text{ is irrational} \end{cases}$ in $[0,1]$

then $g(x)=f(x)f(1-x)$ is continuous everywhere.

7. Let $f(x)=\lim_{n\to\infty}\frac{x^{2n}+3}{x^{2n}+1}$, $x\in[-2,2]$. Test for the continuity of *f* in $[-2,2]$.

8. A function $f:[0,1]\to\mathbb{R}$ is continuous on $[0,1]$ and f assumes only rational values on [0, 1]. Prove that *f* is constant.

9. $f:[0,2]\to\mathbb{R}$ be continuous on $[0,2]$ and $f(0)=f(2)$ prove there exists *a* point *c* in [0, 1] such that $f(c)=f(c+1)$

10. Prove that $\cos x = x^2$ for some $x\in\left(0,\frac{\pi}{2}\right)$

11. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases} 0, \text{ if } x=0 \text{ or } x \text{ is irrational} \\ \frac{1}{q^3}, \text{ if } x=\frac{p}{q} \text{ where } p\in\mathbb{Z}, q\in\mathbb{N} \text{ and } gcd(p,q)=1 \end{cases}$$

show that *f* is differentiable at 0 and $f'(0)=0$

(Hints : For $x\neq 0$, $0\le\left|\frac{f(x)}{x}\right|\le|x|^2$)

12. $f:\mathbb{R}\to\mathbb{R}$ satisfies the condition $|f(x)-f(y)|\le|x-y|^{\alpha}$ when $\alpha>1$ for all $x,y\in\mathbb{R}$. Show that *f* is constant.

13. Prove that the equation $(x-1)^3+(x-2)^3+(x-3)^3+(x-4)^3=0$ has only one real root.

14. Prove that between any two real roots of $e^x \sin x+1=0$, there is at least one real root of $\tan x+1=0$.

15. Given that $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ & $f''(x)$ exists for all $x\in(a,b)$. If $a<c<b$ and $f(a)=f(b)=0$, prove that there exists a point $\xi\in(a,b)$ such that $f(c)=\frac{1}{2}(c-a)(c-b)f''(\xi)$

16. If $a<c<b, f''$ exists in $(a,b), f$ and f' are continuous at the end points a and b, show that there exists $t\in(a,b)$ such that

$$f(c)=\frac{(b-c)f(a)+(c-a)f(b)}{b-a}+\frac{1}{2}(c-a)(c-b)f''(t)$$

(**Hints :** By second mean value theorem,

$f(a)=f(c)+(a-c)f'(c)+\frac{1}{2}(a-c)^2 f''(\xi)$ for some $\xi\in(a,c)$ &

$f(b)=f(c)+(b-c)f'(c)+\frac{1}{2}(b-c)^2 f''(\eta)$ for some $\eta\in(c,b)$

For ex. 15, take $f(a)=0=f(b)$)

17. Let $f:[a,b]\to\mathbb{R}$ be continuous positive valued function, differentiable in (a,b). Show that there exists $c\in(a,b)$ such that

$$\frac{f(b)}{f(a)}=e^{(b-a).f'(c)/f(c)}$$

(**Hints :** Applying L M V theorem to $F(x)=\ln f(x)$ in [a, b])

18. Let f'' exist in $[0,a]$, $a>0$. If $f(0)=0$ and $0<x\le a$, show that there exists $\xi\in(0,x)$ such that $f'(x)-\frac{f(x)}{x}=\frac{1}{2}xf''(\xi)$

Hence show that $\frac{f(x)}{x}$ is increasing in the above interval if $f''(x) > 0$ & is decreasing if $f''(x) < 0$ for all x.

[Consider $\varphi : [0, a] \to \mathbb{R}$ defined by $\phi(x) = -f(x) + xf'(x) + \frac{1}{2}Ax^2$ where $\varphi(0) = \phi(a)$]

19. Let $f : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$ and be derivable in (a, b). If $f^2(b) - f^2(a) = b^2 - a^2$, show that the equation $f'(x).f(x) = x$ has at least one root in (a, b).

20. An open tank with a square base must have a capacity of v liters, what size will it be if the least amount of tin is used.

21. On the curve $y = \frac{1}{1+x^2}$, find a point at which the tangent forms with the x-axis the greatest (in absolute value) angle ?

22. Test the following function for increase or decrease :

$$y = \frac{1}{5}x^5 - \frac{1}{3}x^3$$

23. What right triangle of given perimeter $2p$ has the greatest area ?

24. p is the length of perpendicular from the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the normal at a variable point on the ellipse. Show that the greatest value of p is $a - b$.

25. Find the relative extremum points of f defined by

$$f(x) = \frac{x^2}{(1-x)^3}$$

4.8 Further Readings

1. Introduction to Mathematical Analysis — A gupta (Academic Publishers)
2. Mathematical Analysis — S. C. Malik & Arora (Wiley Eastern Limited)
3. Introduction to Real Analysis — S. K. Mapa (Sarat Book Distributors)
4. First Course in Real Analysis — S. K. Mukherjee (Academic Publishers) (second edition)
5. Mathematical Analysis — Shantinarayan (S. Chand & Co.)

Unit-5 □ Miscellaneous Examples & Exercises

5.0 Objectives

The main objective of the unit is to prevent various Examples and exercises of unit 1, 2 and 3. Also the solutions of each problem have also been given.

5.1 Unit-1

Problems :

1. Let $f(x) = \frac{1}{x^2},\ x \neq 0, x \in \mathbb{R}$

(a) Determine $f(E)$ where $E = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$

(b) Determine $f^{-1}(G)$ where $G = \{x \in \mathbb{R} : 1 \leq x \leq 4\}$

2. Let $g(x) = x^2$ and $f(x) = x + 2$ for $x \in \mathbb{R}$, and let h be the composite function $h = g \circ f$.

(a) Find $h(E)$ where $E = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$

(b) Find $h^{-1}(G)$ where $G = \{x \in \mathbb{R} : 0 \leq x \leq 4\}$

3. Show that if $f : A \rightarrow B$ and E, F are subsets of A then $f(E \cup F) = f(E) \cup f(F)$ and $f(E \cap F) \subseteq f(E) \cap f(F)$

4. Show that if $f : A \to B$ and G, H are subsets of B, then $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$ and $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$

5. Show that the function f defined by $f(x) = \dfrac{x}{\sqrt{x^2+1}}$, $x \in \mathbb{R}$ is a bijection of $\mathbb{R}$ onto $\{y : -1 < y < 1\}$.

6. For $a, b \in \mathbb{R}$ with $a < b$, find an explicit bijection of $A = \{x : a < x < b\}$ onto $B = \{y : 0 < y < 1\}$

7. Let $f : A \to B$ and $g : B \to C$ be functions.

(a) Show that if $g \circ f$ is injective, f is injective

(b) Show that if $g \circ f$ is surjective, g is surjective.

8. Let f, g be functions such that $(g \circ f)$ $(x) = x$ for all $x \in D(f)$ and $(f \circ g)$ $(y) = y$ for all $y \in D(g)$. Prove that $g = f^{-1}$.

9. Suppose that f and g are real-valued functions with common domain $D(\subset \mathbb{R})$. Assume that f and g are bounded.

Then (a) if $f(x) \le g(x) \forall x \in D$, then sup $f(D)) \le$ sup g (D)

(b) if $f(x) \le g(y) \forall x, y \in D$, then sup $f(D) \le$ inf g (D)

10. Let X be a nonempty set, and let f and g be defined on X and bounded. Show that

$$\sup\{f(x) + g(x) : x \in X\} \le \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$

and $$\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} \le \inf\{f(x) + g(x) : x \in X\}$$

11. Let $X = Y = \{x \in \mathbb{R} : 0 < x < 1\}$. Define $h : X \times Y \to \mathbb{R}$ by $h(x, y) = 2x + y$

(a) For each $x \in X$, find $f(x) = \sup\{h(x, y) : y \in Y\}$, then find $\inf\{f(x) : x \in X\}$

(b) For each $y \in Y$, find $g(y) = \inf\{h(x, y) : x \in X\}$ then find $\sup\{g(y) : y \in Y\}$. Compare with the result found in part (*a*).

Solutions of problems :

2. $h = gof : \mathbb{R} \to \mathbb{R}$ is defined by

$$h(x) = (g \circ f)(x)$$

$$= g(f(x))$$

$$= g(x+2)$$

$$= (x+2)^2$$

$$= x^2 + 4x + 4$$

(a) $0 \le x \le 1$

$$\Rightarrow 2 \le x + 2 \le 3$$

$$\Rightarrow 4 \le (x+2)^2 \le 9$$

so $h(E) = \{y \in \mathbb{R} : 4 \le y \le 9\}$

(b) $0 \le (x+2)^2 \le 4$

$$\Rightarrow -2 \le (x+2) \le 2$$

$$\Rightarrow -4 \le x \le 0$$

so $h^{-1}(G) = \{x \in \mathbb{R} : -4 \le x \le 0\}$

where $G = \{x \in \mathbb{R} : 0 \le x \le 4\}$

5. For $x, y \in \mathbb{R}$, let $f(x) = f(y)$

$$\Rightarrow \frac{x}{\sqrt{1+x^2}} = \frac{y}{\sqrt{1+y^2}}$$

$$\Rightarrow \frac{x^2}{1+x^2}=\frac{y^2}{1+y^2}$$

$$\Rightarrow x^2+x^2y^2=y^2+x^2y^2$$

$$\Rightarrow (x+y)(x-y)=0$$

Then $(x+y)(x-y)=0$. But $x+y\neq 0$ for then $y=-x$ & $\frac{x}{\sqrt{(1+x^2)}}=\frac{y}{\sqrt{(1+y^2)}}$ does not stand.

If possible let $x\neq y$

as $\sqrt{1+x^2}, \sqrt{1+y^2}$

both are positive and *x*, *y* both are not zero.

So we arrive at a contradiction.

$\therefore f$ is one to one

$-1<y<1$

$\Rightarrow 0\leq y^2<1$

$\Rightarrow 0<1-y^2\leq 1$

For $x=\frac{y}{\sqrt{1-y^2}}$, $f(x)=\frac{\frac{y}{\sqrt{1-y^2}}}{\sqrt{1+\frac{y^2}{1-y^2}}}=\frac{\frac{y}{\sqrt{1-y^2}}}{\frac{1}{\sqrt{1-y^2}}}=y$

$\therefore f:\mathbb{R}\to\{y\in\mathbb{R}:-1<y<1\}$ is onto. Thus f is a bijecton of $\mathbb{R}$ onto $\{y\in\mathbb{R}:-1<y<1\}$.

and $f^{-1}(x)=\frac{x}{\sqrt{1-x^2}}, -1<x<1$

7. Let for $a, b \in A$ so that

$f(a) = f(b) \Rightarrow g(f(a)) = g(f(b))$

$\Rightarrow (g \circ f)(a) = (g \circ f)(b)$ (As $g \circ f$ is injective)

$\Rightarrow a = b$

Thus f is injective

Let $y \in C$. As $g \circ f : A \to C$ is surjective, there exists $x \in A$ such that $(g \circ f)(x) = y$ or $g(f(x)) = y$

$\Rightarrow$ corresponding to y of C, $\exists\, f(x) \in B \Rightarrow g$ is surjective.

8. Let $y \in D(g), f(g(y)) = (f \circ g)(y) = y$

$\therefore f : D(f) \to D(g)$ is bijective and $f(g(x)) = x \forall x \in D(g)$

therefore, $f^{-1} : D(g) \to D(f)$ is given by $f^{-1}(x) = g(x) \forall x \in D(g)$

Thus, $g = f^{-1}$.

9. (a) $f(x) \leq g(x) \forall x \in D$ so, $f(x) \leq g(x) \leq \sup\{g(x)\}, x \in D$

Let $\lambda = \sup f(D)$, $\mu = \sup g(D)$

$\therefore f(x) \leq \mu \forall x \in D$

If possible let $\lambda > \mu$ choose $\varepsilon = \dfrac{\lambda - \mu}{2}$.

Then there is $x \in D$ s.t. $f(x) > \lambda - \varepsilon$

$= \lambda - \left(\dfrac{\lambda - \mu}{2}\right)$

$= \dfrac{2\lambda - \lambda + \mu}{2}$

$= \dfrac{\lambda + \mu}{2}$

$$f(x)-\mu > \frac{\lambda+\mu-2\mu}{2} = \frac{\lambda-\mu}{2} > 0$$

$\Rightarrow f(x) > \mu$ contradiction. Therefore, $\sup f(D) \le \sup g(D)$

(b) $f(x) \le g(y) \forall x, y \in D$. Fix $a\ y_\circ \in D$. Then $f(x) \le g(y_\circ) \forall x \in D$

Now make $y_\circ$ arbitrary. If possible let Inf g (D) < sup f (D)

Take $\varepsilon = \dfrac{\sup f(D) - \inf g(D)}{2} > 0$

Then there exists $a, b \in D$ such that

$\sup f(D) - \varepsilon < f(a)$ and $g(b) < \inf g(D) + \varepsilon$

$$\Rightarrow \sup f(D) - \frac{\sup f(D) - \inf g(D)}{2} < f(a)$$

$$\Rightarrow \frac{\sup f(D) + \inf g(D)}{2} < f(a) \ldots (1)$$ and

$$g(b) < \frac{\inf g(D) + \sup f(D)}{2} \ldots (2)$$

But $f(a) \le g(b) \ldots (3)$

From (1), (2), (3) we have

$$\frac{\sup f(D) + \inf g(D)}{2} < \frac{\inf g(D) + \sup f(D)}{2}$$

which is a contradiction.

Some solved problems on Limits.

1. Show that for $f(x) = [x]$, $\lim\limits_{x \to 0} [x]$ does not exist.

In an arbitrary neighbourhood of 0, say $N(0,1)$

$$f(x) = -1 \quad \text{if } -1 < x < 0$$
$$= 0 \quad \text{if } 0 \le x < 1$$

Let us consider two sequences $\{x_n\}_n$ and $\{y_n\}_n$ in $N(0,1)$ defined by $x_n = \frac{1}{n+1}, y_n = \frac{-1}{n+1}, n \in \mathbb{N}$. Then the sequences $\{x_n\}_n$ and $\{y_n\}_n$, converge to 0. But the sequence $\{f(x_n)\}$ is {0, 0, 0,...} This converges to 0, and the sequence $\{f(y_n)\}_n$ is {–1, –1, –1,...}. This converges to –1.

$\therefore \lim_{x \to 0}[x]$ does not exist.

2. $\lim_{x \to 0}$ sgn x does not exist.

Let $f(x) = \text{sgn}\, x$.

Then $f(x) = 1$ for $x > 0$

$= 0$ for $x = 0$

$= -1$ for $x < 0$

Domain D of f is $\mathbb{R}$. 0 is an accumulation point of D. Let us consider two sequences $\{x_n\}_n$ in $\mathbb{R}$ and $\{y_n\}_n$ in $\mathbb{R}$ defined by $x_n = \frac{1}{n}$, $y_n = \frac{-1}{n}$, $n \in \mathbb{N}$.

Then $\lim_{n \to \infty} x_n = 0 = \lim_{n \to \infty} y_n$

Also, $f(x_n) = 1 \;\&\; f(y_n) = -1 \; \forall \, n \in \mathbb{N}$

Therefore, $\lim_{n \to \infty} f(x_n) = 1$, $\lim_{n \to \infty} f(y_n) = -1$ which are different

$\therefore \lim_{x \to 0}$ sgn x does not exist.

3. Show that the following limits do not exist :

(a) $\lim_{x \to 0} \frac{1}{x^2} (x > 0)$

(b) $\lim_{x \to 0} (x + \text{sgn}(x))$

4. Use either $\varepsilon-\delta$ definition of limit or the sequential criterion for limits, to establish.

(a) $\lim\limits_{x\to 1}\dfrac{x}{1+x}=\dfrac{1}{2}$

(b) $\lim\limits_{x\to 0}\dfrac{x^2}{|x|}=0$

5. $f(x)=\operatorname{sgn} x$ Examine if $\lim\limits_{x\to 0+} f(x)$ and $\lim\limits_{x\to 0-} f(x)$ exist.

Here the domain D of f is $\mathbb{R}$

Let $D_1=D\cap(0,\infty)$ and $D_2=D\cap(-\infty,0)$. 0 is an accumulation point of both D_1 and D_2.

$f(x)=1\ \forall\ x\in D_1$ and $f(x)=-1\ \forall\ x\in D_2$

Therefore $\lim\limits_{x\to 0+} f(x)=1$ and $\lim\limits_{x\to 0-} f(x)=-1$

6. $f(x)=e^{\frac{1}{x}}$.

Examine if $\lim\limits_{x\to 0+} f(x)$ and $\lim\limits_{x\to 0-} f(x)$ exist.

Here the domain D of f is $R\setminus\{0\}$. Let $D_1=D\cap(0,\infty)$ and $D_2=D\cap(-\infty,0)$. 0 is an accumulation point of D_1 and D_2. f is unbounded on $N(0)\cap D_1$ for any neighbourhood $N(0)$ of 0. Therefore $\lim\limits_{x\to 0+} f(x)$ does not exist.

We have $e^t>t>0\ \forall\ t>0$. Take $t=-\dfrac{1}{x}, x<0$ we have $e^{-\frac{1}{x}}>-\dfrac{1}{x}>0$, and this imples $0<e^{-\frac{1}{x}}<-x\ \forall x<0$

By Sandwich Theorem, $\lim\limits_{x\to 0-} f(x)=0$

7. Show that $\lim\limits_{x\to 0} f(x)=\infty$, where $f(x)=\dfrac{1}{x^2}$

In every neighbouhood of 0. f is unbounded above. Let us choose $G > 0$. Then $f(x) > G\ \forall\ x$ satisfying $x < \frac{1}{\sqrt{G}},\ x \neq 0$

That is, $f(x) > G\ \forall\ x \in N'(0, \delta)$ where $\delta = \frac{1}{\sqrt{G}}$.

Therefore $\lim_{x \to 0} f(x) = \infty$

8. Examine if $\lim_{x \to \frac{\pi}{2}} \tan x$ exists.

Let $f(x) = \tan x$. The domain D of f is $\mathbb{R} - \left\{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\right\} = \mathbb{R}^*$

$D_1 = D \cap \left(\frac{\pi}{2}, \infty\right)$, $D_2 = D \cap \left(-\infty, \frac{\pi}{2}\right)$ $D_1 \neq \phi$, $D_2 \neq \phi$. Also $\frac{\pi}{2}$ is an acumulation point of both D_1 and D_2. In $\frac{\pi}{2} < x < \pi$, f is monotonic decreasing function unbounded below. Therefore, $\lim_{x \to \frac{\pi}{2}+} f(x) = -\infty$.

In $0 < x < \frac{\pi}{2}$, f is a monotonic increasing function unbounded above. Therefore, $\lim_{x \to \frac{\pi}{2}-} f(x) = \infty$. We conclude that $\lim_{x \to \frac{\pi}{2}} f(x)$ does not exist.

9. Using Cauchy's principle, prove that $\lim_{x \to \infty} \cos x$ does not exist.

Let $f(x) = \cos x, x \in \mathbb{R}$. Here the domain of f is $\mathbb{R}$. Let us choose $\varepsilon = \frac{1}{2}$. In order that $\lim_{x \to \infty} f(x)$ should exist, it is necessary that there exists a positive G such that $|f(a) - f(b)| < \frac{1}{2}$ for every pair of points $a, b > G$.

For a given positive real number G. We can find a natural number K such that $2K\pi > G$

Let $a=(2k+1)\pi$, $b=2k\pi$. Then $a,b>G$ and $f(a)=-1$, $f(b)=1$. Therefore, $|f(a)-f(b)|\not<\varepsilon$ for some pair of points $a,b>G$. This shows that cauchy's condition for the existence of $\lim\limits_{x\to\infty}\cos x$ is not satisfied. Therefore $\lim\limits_{x\to\infty}\cos x$ does not exist.

10. Prove that $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e$.

We have $\lim\limits_{x\to\infty}(1+\frac{1}{x})^x=e$. Let $y=\frac{1}{x}$. As $x\to\infty, y\to 0+$

and $x\to-\infty$, $y\to 0-$

Then $e=\lim\limits_{x\to\infty}(1+\frac{1}{x})^x=\lim\limits_{y\to 0+}(1+y)^{1/y}$...(1)

$e=\lim\limits_{x\to\infty}(1+\frac{1}{x})^x=\lim\limits_{y\to 0-}(1+y)^{1/y}$....(2)

From (1) & (2), $\lim\limits_{y\to 0}(1+y)^{1/y}=e$

Thus, $\lim\limits_{x\to 0}(1+x)^{1/x}=e$

1. Use sequential criterion for limits to show that following limits do not exist.

(i) $\lim\limits_{x\to 0}\cos\frac{1}{x^2}$

(ii) $\lim\limits_{x\to\infty}x^{1+\sin x}$

2. $f(x)=x, \quad x\in Q$

$\quad =2-x, \ x\in\mathbb{R}\setminus Q$

Show that (i) $\lim\limits_{x\to 1}f(x)=1$, (ii) $\lim\limits_{x\to c}f(x)$ does not exist, if $c\neq 1$.

3. Show that the following limits do not exist

(i) $\lim_{x\to 0} \dfrac{1}{1+e^{1/x}}$

(ii) $\lim_{x\to 0} \dfrac{2x+|x|}{2x-|x|}$

4. Evaluate the limits

(i) $\lim_{x\to 0+} \sqrt{x-[x]}$, $\lim_{x\to 0-} \sqrt{x-[x]}$

(ii) $\lim_{x\to 0+} x\left[\dfrac{1}{x}\right]$, $\lim_{x\to 0-} x\left[\dfrac{1}{x}\right]$

(iii) $\lim_{x\to 0+}\left[\dfrac{\sin x}{x}\right]$, $\lim_{x\to 0-}\left[\dfrac{\sin x}{x}\right]$

5. Evaluate the limits

(i) $\lim_{x\to\infty} \dfrac{x^2+3x}{x^2+x+1}$

(ii) $\lim_{x\to\infty} \dfrac{\sin x}{x+\cos x}$

(iii) $\lim_{x\to\infty}\left(\sqrt[3]{x+1}-\sqrt[3]{x}\right)$

(iv) $\lim_{x\to 3}\left([x]-\left[\dfrac{x}{3}\right]\right)$

5.2 Unit-2

1. A function $f:\mathbb{R}\to\mathbb{R}$ is continuous on $\mathbb{R}$ and $f(x+y)=f(x)+f(y)\ \forall\ x, y\in\mathbb{R}$. If $f(1)=k$, prove that $f(x)=kx\ \ \forall\ x\in\mathbb{R}$. Also show that f is uniformly continuous on $\mathbb{R}$.

Take $x=y=0$, We have $f(0)=2f(0)\Rightarrow f(0)=0$.....(i)

Take $y=-x,\quad f(x)+f(-x)=0 \Rightarrow f(-x)=-f(x)\ldots\ldots(ii)$

Let x be a positive integer.

Then $f(x) = f(1+1+\ldots+1)$

$= f(1)+f(1)+\ldots\ldots+f(1)$ (x times)

$= x\,f(1)$

$= kx$ if x be a positive integer......(iii)

Let x be a negative integer,

let $x=-y,\ y>0$

$f(x)=f(-y)=-f(y)(by\text{ ii})=-ky$

$\Rightarrow f(x)=kx$, if x be negative integer.

So, $f(x)=kx$ if x is a negative integer......(iv)

From (i), (iii) and (iv) if follows that $f(x)=k\,x$ if x is an integer.... (v)

let $x \in Q,\ x=\dfrac{p}{q}$, 'say' $p\in\mathbb{Z},\ q\in\mathbb{N}$

$f(qx)=f(p)=kp$ by (v)

$f(qx)=f(x+\ldots\ldots+x)$

$=f(x)+f(x)+\ldots\ldots+f(x)$ $[q$ times$]$

$=q\,f(x)$

Therefore $q\,f(x)=k\,p$

or, $f(x)=\dfrac{k\,p}{q}=k\,x$

So, $f(x)=k\,x$ if x is a rational number ......(vi)

Let $\alpha\in\mathbb{R}\setminus Q$. Let us consider a sequence of rational points $\{x_n\}_n$ converging to α. Since f is continuous at α, $\lim\limits_{n\to\infty} f(x_n)=f(\alpha)$.

But $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} kx_n$, since $x_n \in Q$

As $\lim kx_n = k\alpha$, it follows that $f(\alpha) = k\,\alpha$

So, $f(x) = k\,x$ if x is an irrational number .... (vii)

From (v), (vi), (vii) $f(x) = kx\ \forall\ x \in \mathbb{R}$

Let $\varepsilon > 0$

and let x_1, x_2 be any two points in $\mathbb{R}$. Choose $\delta = \dfrac{\varepsilon}{|k|+1}$. Such δ depends only on ε.

Then $|f(x_1) - f(x_2)| = |k||x_1 - x_2| < \varepsilon$. This prove that f is uniformly continuous on $\mathbb{R}$.

2. A function f is defined on $\mathbb{R}$ by
$$f(x) = \cos\frac{1}{x},\quad x \neq 0$$
$$= 0,\quad x = 0$$

Prove that f is not continuous at 0.

Let us consider a sequence $\{x_n\}_n$ where $x_n = \dfrac{1}{2\pi n}$, $n \in \mathbb{N}$. Then

$\lim x_n = 0$, $f(x_n) = 1\quad \forall n \in \mathbb{N}$. Therefore, $\lim f(x_n) = 1$.

We have a sequence $\{x_n\}_n$ in $\mathbb{R}$ that converges to 0 but $\lim f(x_n) \neq f(0)$, proving that f is not continuous at 0.

3. $\lim_{x\to 0+} \sqrt{1+\sqrt{x}} = 1$

Let $f(x) = 1 + \sqrt{x},\ x \geq 0,\ g(x) = \sqrt{x},\ x \geq 0$

Let $A = \{x \in \mathbb{R} : x \geq 0\}$, $f(A) \subset D(g)$

$(g \circ f)(x) = g(f(x)) = \sqrt{1+\sqrt{x}},\ x \geq 0,\ 0 \in A'$ and $\lim_{x\to 0} f(x) = 1$,

$1 \in D(g)$ and g is continuous at 1.

Therefore $\lim_{x \to 0+} \sqrt{1+\sqrt{x}} = \lim_{x \to 0} (g \circ f)(x) = g(1) = 1$

4. $f(x) = \frac{1}{x} \sin \frac{1}{x}, \; x > 0$

$= 0, x = 0$

$\lim_{x \to 0+} f(x) = \infty, \; \lim_{x \to 0-} f(x) = -\infty$. f is discontinuous at 0, 0 is a point of infinite oscillatory discontinuity.

5. $f(x) = x - [x]. \; 0 < x < 2$

$f(x) = x, \; 0 < x < 1$

$= x - 1, \; 1 \le x < 2$

Here $\lim_{x \to 1-} f(x) = 1, \; \lim_{x \to 1+} f(x) = 0, \; f(1) = 0$. f is discontinuous at 1. 1 is a point of jump discontinuity.

Total jump of f at $1 = f(1+0) - f(1-0) = 0 - 1 = -1$

Problems on Chapter - 2

1. Determine the points of continuity of the functions

(a) $g(x) = x[x]$

(b) $k(x) = \left[\frac{1}{x}\right], x \ne 0$

2. $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let $S = \{x \in \mathbb{R} : f(x) = 0\}$ be the "zero set" of f. If $\{x_n\}$ is in S and $x = \lim x_n$, show that $x \in S$.

3. Suppose that $f(r) = 0 \quad \forall \; r \in Q$. Prove that $f(x) = 0 \; \forall \; x \in \mathbb{R}$

4. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 2x$ for $x \in Q$, $g(x) = x + 3, \; x \in \mathbb{R} \setminus Q$. Find points at which g is continuous.

5. Let $g : \mathbb{R} \to \mathbb{R}$ satisfy the relation $g(x+y) = g(x)\, g(y) \; \forall \; x, y \in \mathbb{R}$. Show that if g is continuous at $x = 0$, then g is continuous at every point of $\mathbb{R}$. Also if we have $g(a) = 0$ for some $a \in \mathbb{R}$, then $g(x) = 0 \; \forall \; x \in \mathbb{R}$. Also show that if $g(x) \ne 0$ for any x, then $g(x) = a^x$ where $a > 0, a \ne 1$.

6. Let f be defined by $f(x) = \sin\frac{1}{x}, x \neq 0$ and $f(0) = 0$. Prove that f has the intermediate value property on $[-1, 1]$.

7. Let f be defined on an interval I and suppose that f is one-to-one on I.

(a) Give an example to show that f may not be monotone on I.

(b) Give an example to show that f may not be monotone on any subinterval of I.

(c) Suppose that f is continuous on I. Prove that f is monotone on I.

(d) Suppose that f has the intermediate value property on I. Prove that f is monotone on I.

8. Find the point of discontinuity of the functions.

(i) $f(x) = [\sin x], \ x \in \mathbb{R}$

(ii) $f(x) = (-1)^{[x]}, \ x \in \mathbb{R}$

9. Examine the nature of discontinuity of f at 0.

(i) $f(x) = \frac{1}{\sqrt{x}}, \ x > 0$

$= 0 \quad x = 0$

(ii) $f(x) = \frac{\sin x}{\sqrt{x}}, \ x \neq 0$

$= 0 \quad x = 0$

(iii) $f(x) = \cos\frac{1}{x}, \ x \neq 0$

$= 0 \quad x = 0$

(iv) $f(x) = \frac{1}{x} \sin\frac{1}{x}, x \neq 0$

$= 0, \quad x = 0$

10. Show that f is piecewise continuous on the interval I

(i) $f(x) = [x], I = [0, 3]$

(ii) $f(x) = \text{sgn}\, x, I = [-2, 2]$

(iii) $f(x) = x - [x], I = [0, 3]$

11. Prove that the following functions are uniformly continuous on the indicated interval.

(i) $f(x) = \sqrt{x}$ $on [1, \infty)$

(ii) $f(x) = \dfrac{1}{1+x^2},\ x \in \mathbb{R}$

(iii) $f(x) = x \sin\dfrac{1}{x},\ x \neq 0$

$= 0, \qquad x = 0$ on $[-1, 1]$

(iv) $f(x) = \tan x$ on $[a, b]$

where $-\dfrac{\pi}{2} < a < b < \dfrac{\pi}{2}$

12. $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and let $f(a) < g(a),\ f(b) > g(b)$. Show that there exists a point $c \in (a, b)$ such that $f(c) = g(c)$.

5.3 Unit-3

Some solved problems on Chapter - 3

1. Let $f : [0, 3] \to \mathbb{R}$ be defined by

$f(x) = x,\ 0 \le x \le 1$

$= 2 - x^2,\ 1 < x < 2$

$= x - x^2,\ 2 \le x \le 3$. Find the derivative function f' and its domain.

$f'(x) = 1$ for $x \in (0, 1)$

$= -2x$ for $x \in (1, 2)$

$=1-2x$ for $x\in(2,\ 3)$

$$\lim_{x\to 0+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0+}\frac{x}{x}=1,\ Rf'(0)=1$$

Hence f is differentiable at 0 and $f'(0)=1$

$$\lim_{x\to 1-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1-}\frac{x-1}{x-1}=1,\ Lf'(1)=1$$

$$\lim_{x\to 1+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1+}\frac{(2-x^2)-1}{x-1}=-\ 2,\ Rf'(1)=-2 \text{ so } Lf'(1)\neq Rf'(1)$$

Hence f is not differentiable at 1.

$$\lim_{x\to 2-}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2-}\frac{2-x^2-(-2)}{x-2}=\lim_{x\to 2-}-(x+2)=-4$$

$Lf'(2)=-4$

$$\lim_{x\to 2+}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2+}\frac{x-x^2+2}{x-2}=\lim_{x\to 2+}{}^{-(x+1)=-3}$$

$\therefore Rf'(2)=-3$ so $Lf'(1)\neq Rf'(2)$

Hence f is not differentiable at 2.

$$\lim_{x\to 3-}\frac{f(x)-f(3)}{x-3}=\lim_{x\to 3-}\frac{x-x^2-(-6)}{x-3}=\lim_{x\to 3-}\frac{x-x^2+6}{x-3}$$

$$=\lim_{x\to 3-}-(x+2)=-5\Rightarrow Lf'(3)=-5$$

Hence f is differentiable at 3 and $f'(3)=-5$

The derived function f' is defined by

$$\begin{aligned} f'(x)&=1, && 0\le x<1\\ &=-2x, && 1<x<2\\ &=1-2x, && 2<x\le 3\end{aligned}$$

2. $f(x)=x^2, x\in[0,\infty)$. f is strictly increasing and continuous on $[0,\infty)$.

Let $I=[0, \infty)$. Then $f(I)=[0,\infty)$

The inverse function g defined by $g(y)=\sqrt{y},\quad y\in[0,\infty)$ is continuous in $[0,\infty)$

f is differentiable on $[0,\infty)$ and $f'(x)=2x,\ x\in[0,\infty)$

$f'(x)\neq 0$ *on* $(0, \infty)$. Let $I_1=(0,\infty)$. Then $f(I_1)=(0, \infty)$.

Hence $g'(y)$ exists $\forall\, y\in(0, \infty)$ and $g'(y)=\dfrac{1}{f'(x)}=\dfrac{1}{2x}=\dfrac{1}{2g(y)}$

$=\dfrac{1}{2\sqrt{y}},\ y\in(0, \infty)$

3. A function $f:\mathbb{R}\to\mathbb{R}$ is defined by $f(0)=0$ and $f(x)=0$ if $x\in\mathbb{R}\setminus Q$

$f(x)=\dfrac{1}{q},\ x=\dfrac{p}{q}$ where $p\in\mathbb{Z},\ q\in\mathbb{N}$ and $\gcd(p,q)=1$

Show that f is not differentiable at 0.

$\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0}\dfrac{f(x)}{x}$. Let $\phi(x)=\dfrac{f(x)}{x}$. Let $\{x_n\}_n$ be the sequence of rational points converging to 0 where $x_n=\dfrac{1}{n},\ n\in\mathbb{N}$. Then $\lim\limits_{n\to\infty}\phi(x_n)=\lim\limits_{n\to\infty} n.\dfrac{1}{n}=1$

Let $\{y_n\}_n$ be a sequence of irrational points converging to 0.

$\lim\limits_{n\to\infty}\phi(y_n)=\lim\limits_{n\to\infty}\dfrac{f(y_n)}{y_n}=0$, since $f(y_n)=0\ \forall\, n\in\mathbb{N}$.

Therefore, $\lim\limits_{x\to 0}\phi(x)$ does not exist, since for two sequences $\{x_n\}_n$ and $\{y_n\}_n$ both converging to 0, the sequences $\{\phi(x_n)\}_n$ and $\{\phi(y_n)\}_n$ converge to two different limits.

$\therefore f$ is not differentiable at 0.

4. Let $f:[-1,1]\to\mathbb{R}$ be defined by $f(x)=0, x\in[-1,0]$, $f(x)=1, x\in(0,1]$

Does there exist a function g such that $g'(x)=f(x), x\in[-1,1]$?

If possible, let there exist a function $g:[-1,1]\to\mathbb{R}$ such that

$g'(x)=f(x), x\in[-1,1]$.

Then g is differentiable on $[-1,1]$ and $g'(x)=0, x\in[-1,0]$ and $=1,\ x\in(0,1]$

Since g is differentiable on $[-1,1]$ and $g'(-1)\neq g'(1)$, by Darboux' s theorem g' must assume every real number lying between $g'(-1)$ and $g'(1)$, i. e. between 0 and 1 on $[-1,1]$. But this is not so and therefore g does not exist.

5. Show that functions $\tan^{-1}x=f(x), -\infty<x<\infty$, is uniformly continuous there in and $f'(x)$ is also so.

Let x, y be any pair of points in $(-\infty,\infty)$. By LMV theorem, $\exists\xi\in(x,y)$ such that $f(y)-f(x)=(y-x)f'(\xi)\Rightarrow|f(y)-f(x)|=|y-x|\frac{1}{1+\xi^2}\le|y-x|$. Let $\varepsilon>0$ be any number. So $|f(y)-f(x)|\le|y-x|<\varepsilon$ whenever $|y-x|<\delta, \delta=\varepsilon$.

$\Rightarrow f$ is uniformly continuous on $(-\infty,\infty)$.

Again $|f'(x)-f'(y)|=|x-y|f''(\xi)$ for some $\xi\in(x,y)$ (by LMV theorem)

$$|f''(\xi)|=\left|\frac{2\xi}{\left(\xi^2+1\right)^2}\right|.$$

Note : That $\frac{1+\xi^2}{2}\ge|\xi|\Rightarrow\frac{2|\xi|}{1+\xi^2}\le 1$. Also $\frac{1}{1+\xi^2}\le 1$

consequently, $|f'(x)-f'(y)|\le|x-y|<\varepsilon$ whenever $|x-y|<\delta$

$\Rightarrow f'$ is uniformly continuous on $(-\infty,\infty)$.

Indeterminate form

In the process of examining the existence of limit of functions in $\mathbb{R}$ and in its determination we are very often faced with limits of following forms :

$$\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, \infty^0, 1^\infty.$$

These forms are generally known as 'Indeterminate forms'. Usually all the above forms can be reformulated to give rise to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

French mathematician G. F. L. Hopital (1661-1704) gave a method for computing such limits (provided they exist).

L' Hopital's Rule,

Result-1 : Let $f, g : [a,b] \to \mathbb{R}$ be both continuous in $[a, b]$, differentiable in (a,b) such that $f(a) = 0 = g(a)$ & $g(x) \neq 0$, $g'(x) \neq 0$ in $a < x < b$. Then,

(i) if $\lim\limits_{x\to a+} \frac{f'(x)}{g'(x)} = l(\in R)$, then $\lim\limits_{x\to a+} \frac{f(x)}{g(x)} = l$

(ii) if $\lim\limits_{x\to a+} \frac{f'(x)}{g'(x)} = \infty$, then $\lim\limits_{x\to a+} \frac{f(x)}{g(x)} = \infty$.

Proof of (i) $\lim\limits_{x\to a+} \frac{f'(x)}{g'(x)} = l \Rightarrow$ corresponding to arbitrary $\varepsilon > 0, \exists\, \delta > 0$, $0 < \delta < b - a$, such that

$$\left|\frac{f'(x)}{g'(x)} - l\right| < \varepsilon \text{ whenever } a < x < a + \delta \qquad \dots (1)$$

By hypothesis, Cauchy's M. V. theorem is applicable to $f \,\&\, g$ in $[a,x]$ where $a < x < a + \delta$. $\exists\, \xi_x, a < \xi_x < x$, such that

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi_x)}{g'(\xi_x)} \;\&\text{ so } \frac{f(x)}{g(x)} = \frac{f'(\xi_x)}{g'(\xi_x)} \qquad \dots(2)$$

(1) & (2) $\Rightarrow \left|\frac{f(x)}{g(x)} - l\right| < \varepsilon$ whenever $a < x < a + \delta$

$$\Rightarrow \lim_{x\to a+} \frac{f(x)}{g(x)} = l$$

Proof of (ii) : $\lim_{x\to a+}\frac{f'(x)}{g'(x)}=\infty \Rightarrow$ corresponding to any $G > 0$, as large as we please, $\exists\,\delta, 0<\delta<b-a,$ such that

$$\frac{f'(x)}{g'(x)}>G \text{ whenever } a<x<a+\delta \qquad \ldots (1)$$

Again Cauchy's M V theorem is applicable to f, g in $[a, x]$, $a<x<a+\delta$.

so $\exists\, \eta_x, a<\eta_x<x,$ such that

$$\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f'(\eta_x)}{g'(\eta_x)}\Rightarrow\frac{f(x)}{g(x)}=\frac{f'(\eta_x)}{g'(\eta_x)} \quad \ldots (2)$$

By (1) & (2), $\frac{f(x)}{g(x)}>G$ whenever $a<x<a+\delta$

$$\Rightarrow \lim_{x\to a+0}\frac{f(x)}{g(x)}=\infty.$$

Simple Ilustrations :

(1) $\lim_{x\to 0+}\frac{\sin x}{\sqrt{x}}\left(\frac{0}{0}\right)=\lim_{x\to 0+}\frac{\cos x}{\frac{1}{2\sqrt{x}}}=\lim_{x\to 0+}2\sqrt{x}\cos=0$

(2) $\lim_{x\to 0}\frac{1-\cos x}{x^2}\left(\frac{0}{0}\right)=\lim_{x\to 0}\frac{\sin x}{2x}=\frac{1}{2}.$

(3) Evaluate $\lim_{x\to\infty}\left[x-\sqrt[n]{(x-a_1)\ldots\ldots(x-a_n)}\right]$ where a_i's are positive rationals.

The limit is of form $\infty-\infty$.

We take $x=\frac{1}{t}$. so $x\to\infty \Leftrightarrow t\to 0+$

The limit is $\lim_{t\to 0}\frac{1-\sqrt[n]{(1-a_1t)(1-a_2t)\ldots(1-a_nt)}}{t}\left(\frac{0}{0}\right)$

$$=\lim_{t\to 0}\frac{-\frac{d}{dt}\left\{\sqrt[n]{(1-a_1t)\ldots(1-a_nt)}\right.}{1}$$

$$= \lim_{t \to 0} \frac{\frac{f(t)}{n}\left[\frac{a_1}{1-a_1 t} + \frac{a_2}{1-a_2 t} + \ldots + \frac{a_n}{1-a_n t}\right]}{1}$$

where $f(t) = \sqrt[n]{\left[(1-a_1 t)(1-a_2 t)\ldots(1-a_n t)\right]}$

So the required limit is $\frac{1}{n}(a_1 + a_2 + \ldots + a_n)$.

(4) Evaluate $\lim_{x \to 0} \left(e^x + x\right)^{\frac{1}{x}}$.

The limit is of form 1^{∞}

Let $y = \left(e^x + x\right)^{\frac{1}{x}}$, so $\log y = \frac{1}{x}\log(e^x + x)$.

Then $\lim_{x \to 0} \log y = \lim_{x \to 0} \frac{\log(e^x + x)}{x} \left(\frac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{1\left(e^x + 1\right)}{e^x + x} = 2$$

So $\lim_{x \to 0} y = e^2$.

Note : Standard limits like $\lim_{x \to 0} \frac{\sin x}{x}$, $\lim_{x \to 0} \frac{\log(1+x)}{x}$, $\lim_{x \to 0} \frac{e^x - 1}{x}$, $\lim_{x \to a} \frac{x^n - a^n}{x - a}$ etc can not be evaluated by L. Hopital's rule.

The reason is that if you apply the above rule to find $\lim_{x \to 0} \frac{\sin x}{x}$, then you are differentiating $\sin x$ w.r.t. x & in order to do the same, you are using the limit $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Result II : Let f, g are differentiable in $[a,b]$, $\lim_{x \to a+} f(x) = \infty = \lim_{x \to a+} g(x)$, $g(x) \neq 0$, $g'(x) \neq 0$ in $a < x < b$, then

(i) if $\lim_{x \to a+} \frac{f'(x)}{g'(x)} = l(\in \mathbb{R})$, then $\lim_{x \to a+} \frac{f(x)}{g(x)} = l$

(ii) if $\lim_{x \to a+} \frac{f'(x)}{g'(x)} = \infty$, then $\lim_{x \to a+} \frac{f(x)}{g(x)} = \infty$

Proof : Let $a<x<c<b$.

Applying C M V theorem to f & g in [a, c], $\exists \xi \in (a,c)$ such that

$$\frac{f(x)-f(c)}{g(x)-g(c)}=\frac{f'(\xi)}{g'(\xi)} \Rightarrow \frac{f(x)}{g(x)}=\frac{f'(\xi)}{g'(\xi)}, \frac{1-\frac{g(c)}{g(x)}}{1-\frac{f(c)}{f(x)}} \quad \text{... (1)}$$

Let $0<\varepsilon<1$ be any number. Then

$\exists\, \delta,\ 0<\delta<b-a$ such that $\left|\frac{f'(x)}{g'(x)}-l\right|<\varepsilon,\ a<x<a+\delta$

we can write $\frac{f'(\xi)}{g'(\xi)}=l+\delta_1$ where $|\delta_1|<l<1$

Keeping chosen c fixed. $\lim\limits_{x\to a+}\frac{1-\frac{g(c)}{g(x)}}{1-\frac{f(c)}{f(x)}}=1$.

Choosing x nearer to a.

$$\frac{1-\frac{g(c)}{g(x)}}{1-\frac{f(c)}{f(x)}}=1+\delta_2 \text{ where } |\delta_2|<\begin{cases}\varepsilon, \text{if } |l|<1\\ \frac{\varepsilon}{|l|}, \text{if } |l|>1\end{cases}$$

Then (i) $\Rightarrow \frac{f(x)}{g(x)}=(l+\delta_1)(1+\delta_2)=l+\delta_1+l\delta_2+\delta_1\delta_2$

$\Rightarrow \left|\frac{f(x)}{g(x)}-l\right|<3\varepsilon$ in above neighbourhood of 'a'

$\Rightarrow \lim\limits_{x\to a+}\frac{f(x)}{g(x)}=l$.

(ii) $\lim\limits_{x\to a+}\frac{f'(x)}{g'(x)}=\infty \Rightarrow$ corresponding to arbitrary large positive number G, $\exists\, \delta$,

$0<\delta<b-a$, such that $\frac{f'(x)}{g'(x)}>G,\ a<x<a+\delta$.

Choosing $c \in (a, a+\delta)$

$$\frac{1-\dfrac{g(c)}{g(x)}}{1-\dfrac{f(c)}{f(x)}} > 1-\frac{1}{G}\left(\text{as } \frac{1-\dfrac{g(c)}{g(x)}}{1-\dfrac{f(c)}{f(x)}} \to 1\right)$$

Then (1) $\Rightarrow \dfrac{f(x)}{g(x)} > G\left(1-\dfrac{1}{G}\right)(=G-1) \Rightarrow \lim\limits_{x\to a+}\dfrac{f(x)}{g(g)} = \infty$.

Simple Illustrations:

(1) $\lim\limits_{x\to\infty}\dfrac{\ln x}{x}\left(\dfrac{\infty}{\infty}\right) = \lim\limits_{x\to\infty}\dfrac{1}{x} = 0$

(2) $\lim\limits_{x\to\infty}\left(e^{-x}.\, x^2\right)\left(\dfrac{\infty}{\infty}\right) = \lim\limits_{x\to\infty}\dfrac{2x}{e^x}\left(\dfrac{\infty}{\infty}\right) = \lim\limits_{x\to\infty}\dfrac{2}{e^x} = 0$

(3) $\lim\limits_{x\to\frac{\pi}{2}+}\dfrac{\log\left(x-\dfrac{\pi}{2}\right)}{\tan x}\left(\dfrac{\infty}{\infty}\right) = \lim\limits_{x\to\frac{\pi}{2}+}\dfrac{\cos^2 x}{x-\dfrac{\pi}{2}}\left(\dfrac{0}{0}\right)$

$$= \lim_{x\to\frac{\pi}{2}+}\frac{-\sin 2x}{1} = -\sin\pi = 0$$

(4) $\lim\limits_{x\to 0+} x^{\sin x}(0^{\circ})$

let $y = x^{\sin x}$. So $\lim\limits_{x\to 0+}\log y = \lim\limits_{x\to 0+}\sin x.\log x\ (0\times -\infty)$

$$= \lim_{x\to 0+}\frac{\log x}{\operatorname{cosec} x}\left(\frac{-\infty}{\infty}\right) = \lim_{x\to 0+}\frac{\dfrac{1}{x}}{-\operatorname{cosec} x\,\cot x}$$

$$= \lim_{x\to 0+}\frac{\sin 2x}{\cos x - x\sin x} = 0 \Rightarrow \lim_{x\to 0+} y = e^{\circ} = 1.$$

(5) $\lim\limits_{x\to 0+}\left(\dfrac{1}{x}\right)^{\tan x}(\infty^{\circ})$

let $y = \left(\frac{1}{x}\right)^{\tan x} \Rightarrow \log y = \tan x \log\left(\frac{1}{x}\right)$

$$\Rightarrow \lim_{x\to 0+} \log y = \lim_{x\to 0+} \tan x\,(-\log x) \quad (0\times\infty)$$

$$= \lim_{x\to 0+} \frac{-\log x}{\cot x}\left(\frac{\infty}{\infty}\right) = \lim_{x\to 0+} \frac{\frac{1}{x}}{-\operatorname{cosec}^2 x} = \lim_{x\to 0+} \frac{\sin^2 x}{x} = \lim_{x\to 0+} \sin 2x = 0$$

so $\lim_{x\to 0+} y = e^0 = 1$.

(6) $\lim_{x\to\infty} \left[\frac{a_1^{\frac{1}{x}} + a_2^{\frac{1}{x}} + \ldots + a_n^{\frac{1}{x}}}{}\right]^{nx}$, a_i's are positive rationals.

let y be the expression mentioned above

$$\lim_{x\to\infty} \log y = \lim_{x\to\infty} \frac{\log\left[\frac{a_1^{\frac{1}{x}} + a_2^{\frac{1}{x}} + \ldots + a_n^{\frac{1}{x}}}{n}\right]}{\frac{1}{n_x}}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to\infty} \left\{ \frac{\frac{n}{a_1^{\frac{1}{x}} + \ldots + a_n^{\frac{1}{x}}} \cdot \frac{1}{n}\left[a_1^{\frac{1}{x}} \log_e a_1 + \ldots + a_n^{\frac{1}{x}} \log_e a_n\right]\left(\frac{-1}{x^2}\right)}{\frac{-1}{nx^2}} \right\}$$

$$= \lim_{x\to\infty} \left[\frac{n}{a_1^{\frac{1}{x}} + \ldots + a_n^{\frac{1}{x}}}\right]\left[a_1^{\frac{1}{x}} \log_e a_1 + \ldots + a_n^{\frac{1}{x}} \log_e a_n\right]$$

$$= \log_e (a_1 a_2 \ldots a_n)$$

So, $\lim_{x\to\infty} y = a_1 a_2 \ldots a_n$.

Problems on Indeterminate form : Evaluate the limits in (1) to (5) :

(1) $\lim\limits_{x\to 0+}\left(x\log\sin^2 x\right)$

(2) $\lim\limits_{x\to 0+}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}$

(3) $\lim\limits_{x\to 0}\dfrac{3^x-2^x}{4^x-3^x}$

(4) $\lim\limits_{x\to a}\left(2-\dfrac{x}{a}\right)^{\tan\frac{\pi x}{2a}}$

(5) $\lim\limits_{x\to 1}\left(\dfrac{x}{x-1}-\dfrac{1}{\ln x}\right)$

(6) Determine the constants *a*, *b*, *c* such that

$$\lim_{x\to 0}\frac{x(a+b\cos x)+c\sin x}{x^5}=\frac{1}{60}.$$

(7) If $\lim\limits_{x\to 0}\left(\dfrac{1+Cx}{1-Cx}\right)^{1/x}=4$, find $\lim\limits_{x\to 0}\left(\dfrac{1-2Cx}{1+2Cx}\right)^{1/x}$.

5.4 Summary

In this unit, we have given various problems and solution of the units 1, 2 and 3.

Unit-6 □ Limit and continuity for function of two variables

Structure

6.0 Objectives

This unit gives

- Some preliminary notion of the distance in $\mathbb{R}^2$, diameter of a set, open and closed sts in $\mathbb{R}^2$.
- Convergence of sequence in $\mathbb{R}^2$.
- The concept of limit and continuity of two variable function.
- Continuity of a function in $\mathbb{R}^2$.

6.1 Introduction

This unit concerned with the calculus of functions whose domains are subsets of $\mathbb{R}^2$. Such functions are frequently called by the name "functions of several variables". The concept extends the idea of a function of a real variable to several variables. There are so many applications of this several variables' functions in geometry, applied mathematics, engineering, natural sciences and economics.

6.2 Preliminaries

[1] The set $\mathbb{R}^2$ and the distance on it

$\mathbb{R}^2$ denotes the set of ordered 2-tuples (x^1, x^2) (or ordered pairs) of real numbers $x^i \in \mathbb{R}$ for $i = 1, 2$, where the notion of distance between the points

$$x_1 = (x_1^1, x_1^2) \; and \; x_2 = (x_2^1, x_2^2)$$

is defined by $d(x_1, x_2) = \sqrt{\left[\sum_{i=1}^{2}\left\{(x_1^i - x_2^i)^2\right\}\right]}$ ....(1)

The function d : defined by (1), obeys the following properties :

(i) $d(x_1, x_2) > 0$ (ii) $d(x_1, x_2) = 0 \iff x_1 = x_2$ (iii) $d(x_1, x_2) = d(x_2, x_1)$

(iv) $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$.

[2] Diameter of a set :

The diameter of a set $E \subset \mathbb{R}^2$ is the quantity

$$d(E) = \underset{x_1, x_2 \in E}{Sup}\, d(x_1, x_2)$$

[3] Bounded set in $\mathbb{R}^2$:

A set $E \subset \mathbb{R}^2$ is bounded if its diameter is finite.

[4] Open and closed sets in $\mathbb{R}^2$:

Definition (1) : For $\delta > 0$, the set $B(a, \delta) = \{x \in \mathbb{R}^2 \mid d(a, x) < \delta\}$ is called the ball with centre $a \in \mathbb{R}^2$ of radius δ or the δ-neighbourhood of the point 'a' in $\mathbb{R}^2$.

In particular, if $(a, b) \in \mathbb{R}^2$ and $\delta > 0$, the set

$$\left\{(x, y) \in \mathbb{R}^2 \mid \sqrt{(x-a)^2 + (y-b)^2} < \delta\right\}$$

is called an open disc of radius δ with centre at (a, b) & is denoted by $N(a, b, \delta)$.

The set $N'((a, b), \delta) \equiv N((a, b), \delta) - \{(a, b)\}$ is called the deleted δ-neighbourhood of (a, b), denoted by

$N'((a, b), \delta)$ as mentioned above.

The set $M \subset \mathbb{R}^2$ is a neighbourhood of (a, b) if and only if there exists $\delta > 0$ such that

$$N((a,b),\delta) \subset M$$

Definition (2) : A set G ($\subset \mathbb{R}^2$) is open in $\mathbb{R}^2$ if every point $x \in G$, there exists a ball $B(x, \delta) \subset G$.

In other words, a set G ($\subset \mathbb{R}^2$) is said to be open if it is a neighbourhood of each of its points.

Examples : (i) $\mathbb{R}^2$ is an open set (ii) Void set is open set (iii) A ball $B(a, b)$ is an open set in $\mathbb{R}^2$

Definition (3) : The set $F \subset \mathbb{R}^2$ is closed in $\mathbb{R}^2$ if its complement

$G = \mathbb{R}^2 \backslash F$ is open in $\mathbb{R}^2$.

Definition (4) : Let $E \subset \mathbb{R}^2$. A point x is interior point of E if some neighbourhood of it is contained in E.

On the otherhand, x is exterior point of E if it is an interior point of the complement of E in $\mathbb{R}^2$.

x is boundary point of E if it is neither an interior point of E nor an exterior point of E.

Definition (5) : A point $a \in \mathbb{R}^2$ is a limit point (accumulation point) of $E \subset \mathbb{R}^2$ if for any neighbourhood $N(a)$ of 'a' the intersection $E \cap N(a)$ is an infinite set.

The union of set E and all its limit points in $\mathbb{R}^2$, is the closure of E in $\mathbb{R}^2$, denoted by $\overline{E}$.

Results : We state the following results without proof :

(1) Intersection of any two open discs is an open set.

(2) The union of any number of open subsets of $\mathbb{R}^2$ is open.

(3) The intersection $\bigcap_{i=1}^{n} G_i$ of a finite number of open sets in $\mathbb{R}^2$ is an open set.

(4) The intersection $\bigcap_{\alpha \in \wedge} F_\alpha$ of the sets of any system $\{F_\alpha : \alpha \in \wedge\}$ of closed sets F_α in $\mathbb{R}^2$ is a closed set in $\mathbb{R}^2$. ($\wedge$: Index set).

(5) The union of finite number of closed sets in $\mathbb{R}^2$ is a closed set in $\mathbb{R}^2$.

(6) Every bounded infinite set S of points in a plane has at least one accumulation point.

Sequence of points : Convergence

Let us consider infinite sequence of points $P_n(x_n, y_n)$ in the plane. The sequence is bounded if a disc can be found containing all the points P_n *i.e.* if there is a point Q and a number M such that the distance $|P_nQ| < M$ for all $n \in \mathbb{N}$.

Examples : The sequence $P_n = \{(-1)^n + \frac{1}{n}, \frac{3}{n^2}\}_n$, $P_n^1 = \left\{3, \left(\frac{-2}{5}\right)^n\right\}_n$ are bounded but the sequence $\{(n^2, n^3)\}_n$ is not bounded. The sequence $\{P_n\}_n$ converges to a point Q $(or, \lim_{n\to\infty} P_n = Q)$ if the sequence of distances $\{\overline{P_nQ}\}_n$ converges to zero. For every $\varepsilon > 0$, there exists positive integer m such that P_n's lie in the ε-neighbourhood of Q for all n > m.

The sequence of points $\{(x_n, y_n)\}_n$ converges to (a, b) if and only if the sequences $\{x_n\}_n$ & $\{y_n\}_n$ converge to a and b respectively (Co-ordinatewise convergence).

6.3 Limit and continuity for function of two variables

Definition :

$u(x, y)$ is a function of independent variables x and y whenever some law f assigns a unique value of u, the dependent variable to each pair of values (x, y) belonging to a certain specified set, the domain of the function. A function $u(x, y)$ thus defines a mapping of a set of points in the xy-plane, the domain of f, on to a certain set of points on the u-axis, the range of f.

Geometrically, a function of two variables represents a surface.

Examples :

[1] Domain of $u(x, y) = \text{Sin}^{-1}\frac{x}{3} + \sqrt{xy}$ is $S_1 \cup S_2$ where

$$S_1 = \{(x, y) \in \mathbb{R}^2 \mid -3 \le x \le 0, y \le 0\} \;\&\; S_2 = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 3, y \ge 0\}$$

[2] Domain of $u(x,y)=\sqrt{1-x^2}+\sqrt{1-y^2}$ is the square formed by the segments of the lines $x=\pm1, y=\pm1,$ including its sides $|x|\le 1, |y|\le 1$.

[3] Domain of $u(x,y)=\sqrt{(x^2-4)}+\sqrt{(4-y^2)}$ is the two strips $x\ge 2, -2\le y\le 2$ and $x\le -2, -2\le y\le 2$.

[4] Domain of $u(x,y)=\sqrt{y\sin x}$ is the strips $2n\Pi\le x\le(2n+1)\Pi, y\ge 0$ and $(2n+1)\Pi\le x\le(2n+2)\Pi,\ y\le 0$ (where n is integer).

[5] Domain of u(x, y) = ln (x^2 + y) is that part of the plane located above the parabola $y = -x^2$.

Definition (Limit of function of two variables)

Let $f:S\to\mathbb{R}$ when $S\subset\mathbb{R}^2$. Let (a, b) be an accumulation point of S. We say that $\lim\limits_{(x,y)\to(a,b)} f(x,y)=l(\in\mathbb{R})$ if for any number $\varepsilon>0$, there exists $\delta > 0$ such that $|f(x, y) - l| < \varepsilon$ whenever $(x, y)\in N'((a,b),\delta)\cap S$.

This limit if exists is known as simultaneous limit or double limit.

Sequential approach : $\lim\limits_{(x,y)\to(a,b)} f(x,y)=l(\in\mathbb{R})$ if and only if for every sequence of points $(x_n, y_n)\to(a, b),$ we have $\lim\limits_{n\to\infty} f(x_n, y_n)=l(\in IR)$.

These two definitions ($\varepsilon-\delta$ approach & sequential approach) are equivalent.

Examples :

[1] $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$

The sequences $\left\{\left(\dfrac{1}{n},\dfrac{1}{n}\right)\right\}_n$ & $\left\{\left(\dfrac{2}{n},\dfrac{1}{n}\right)\right\}_n$ both approach to (0, 0)

$f\left(\dfrac{1}{n},\dfrac{1}{n}\right)=\dfrac{1/n^2}{2/n^2}=\dfrac{1}{2}$ & $f\left(\dfrac{2}{n},\dfrac{1}{n}\right)=\dfrac{2}{5}$. These two are different.

So the limit does not exist.

[2] $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{(x^2+y^2)}}$

Refer to polar co-ordinates $x = r\cos\theta, y = r\sin\theta$. Then,

$$\frac{xy}{\sqrt{(x^2+y^2)}} = \frac{1}{2}r \sin 2\theta.$$

We note that $|f(x,y)-0| \le \frac{1}{2}r\left(=\frac{1}{2}\sqrt{(x^2+y^2)}\right) < \frac{\varepsilon}{2}$ whenever

$0 < \sqrt{(x^2+y^2)} < \delta \;\&\; \delta = \frac{\varepsilon}{2}, \varepsilon > 0$ is arbitrary number.

So, $\lim_{(x,y)\to(0,0)} f(x,y) = 0.$

Theorem : (Necessary condition for the existence of double limit)

Let $f : S \to \mathbb{R}$ where $S \subset \mathbb{R}^2$ and (a, b) be an accumulation point of S.

If $\lim_{(x,y)\to(a,b)} f(x,y) = L(\in \mathbb{R})$, then $f(x, \varphi(x)) \to L$ as $x \to a$, where φ is a real valued function of one variable x such that $(x, \varphi(x) \in S$

$x \in D_\varphi$ and $\varphi(x) \to b \text{ as } x \to a$.

Note that in a plane (x, y) may approach to (a, b) through infinitely many paths, strictly within the domain. The genesis of the above theorem is that limit φ is independent of all such paths leading to (a, b).

Proof : Given $\lim_{(x, y) \to (a, b)} f(x, y) = L(\in \mathbb{R})$, Let $\varepsilon > 0$ be any number.

Corresponding to ε, there exists $\delta > 0$ such that

$|f(x,y) - L| < \varepsilon$ whenever $0 < |x-a| < \delta, 0 < |y-b| < \delta$ (1)

Again $\lim_{x\to a} \varphi(x) = b$, corresponding to above δ, there exists $\eta > 0$

such that $|\varphi(x) - b| < \delta$ whenever $0 < |x-a| < \eta$ (2)

Let $\rho = \min\{\delta, \eta\}$.

Hence by (1) and (2), we have $|f(x, \varphi(x)) - L| < \varepsilon$ wherever $0 < |x - a| < \rho$

Consequently, $\lim_{x \to a} f(x, \varphi(x)) = L$

Remarks : If there be two functions $\varphi_1(x)$ & $\varphi_2(x)$ such that

$\lim_{x \to a} f(x, \varphi_1(x)) \neq \lim_{x \to a} f(x, \varphi_2(x))$

where $(x, \varphi_1(x))$ & $(x, \varphi_2(x)) \in S$ for each $x \in D_{\varphi_i} (i = 1, 2)$ & as $x \to a$,

$\varphi_i(x) \to b \ (i = 1, 2)$, then $\lim_{(x,y) \to (a,b)} f(x, y)$ does not exist.

Examples : $f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & x \neq y \\ 0, & x = y. \end{cases}$

Let $(x, y) \to (0, 0)$ along the path $x - y = mx^3$. Then

$$\frac{x^3 + y^3}{x - y} = \frac{1 + (1 - mx^2)^3}{m} \to \frac{2}{m} \text{ as } x \to 0, \frac{2}{m} \text{ is different for different m.}$$

So, $\lim_{(x,y) \to (0,0)} f(x, y)$ does not exist.

Repeated limits :

Let $f : S \to \mathbb{R}$ where $S \subset \mathbb{R}^2$ and (a, b) be an accumulation point of S.

Let $\lim_{(x \to a)} f(x, y)$ exist, then it is function of y, say $\varphi(y)$.

Let $\lim_{y \to b} \varphi(y)$ exist & $= \lambda (\in \mathbb{R})$, then $\lim_{y \to b} \lim_{x \to a} f(x, y) = \lambda$ ....(1)

Let $\lim_{y \to b} f(x, y)$ exist & it is then function of x, say $\psi(x)$.

Let $\lim_{x \to a} \psi(x)$ exist & $= \mu (\in \mathbb{R})$, then $\lim_{x \to a} \lim_{y \to b} f(x, y) = \mu$ ....(2)

(1) and (2) are known as 'Repeated limits'

So questions arise regarding the existence of Repeated limits, whether their existence ensure the existence of double limit & conversely etc.

In this connection, let us consider first the following examples :

(1) Let $f(x, y) = \begin{cases} xSin\frac{1}{y} + ySin\frac{1}{x}, xy \neq 0 \end{cases}$

Let ε > 0 be any number,

$$|f(x,y) - 0| = |xSin\frac{1}{y} + ySin\frac{1}{x}| \leq |x| + |y| < \varepsilon \text{ whenever}$$

$$0 < |x - 0| < \delta, 0 < |y - 0| < \delta, \delta \text{ correspond to } \varepsilon, \text{ say } \frac{\varepsilon}{2}$$

So, $\lim_{(x,y)\to(0,0)} f(x,y)$ exists & is 0.

But $\lim_{t\to 0} Sin\frac{1}{t}$ does not exist, so neither $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ nor $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$ exists.

This example illustrates that only the existence of double limit at a point does not ensure the existence of repeated limits.

(2) $f(x,y) = \dfrac{\text{Sin } x + \text{Sin } 2y}{\tan 2x + \tan y}$

Keeping y fixed, let $x \to 0$, then we take $y \to 0$.

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \frac{1}{2}$$

On the other hand, first keeping x fixed, let $y \to 0$. Then we take $x \to 0$.

We get $\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \frac{1}{2}$

So both repeated limits exist though they are unequal.

For consideration of double limit, let $(x, y) \to (0, 0)$ along $y = x$.

$$\frac{\sin x + \sin 2y}{\tan 2x + \tan y} = \frac{\sin x + \sin 2x}{\tan 2x + \tan x} = \frac{2\sin\frac{3x}{2}\cos\frac{x}{2}\cos x\cos 2x}{2\sin\frac{3x}{2}\cos\frac{3x}{2}}$$

$$= \frac{\cos\frac{x}{2}\cos x\cos 2x}{\cos\frac{3x}{2}} \to 1 \text{ as } x \to 0.$$

Next we consider the path y = 2x.

$$\frac{\sin 2x + \sin 2y}{\tan 2x + \tan y} = \frac{\sin x + \sin 4x}{2\tan 2x} = \frac{2\sin\frac{5x}{2}\cos\frac{3x}{2}\cos 2x}{2\sin 2x}$$

$$= \frac{1}{2}.\frac{\sin\frac{5x}{2}}{\frac{5x}{2}}.\frac{x}{\sin x}.\frac{5}{2}.\cos\frac{3x}{2}.\cos 2x.\frac{1}{\cos x} = \frac{5}{4} \text{ as } x \to 0$$

So the double limit does not exist.

So existence of repeated limits $\not\Rightarrow$ existence of double limit.

Also if g(x,y) = $\frac{xy}{x^2+y^2}, x^2+y^2 \neq 0,$ we note that both the repeated limits exist and are equal but the double limit does not exist.

So a question arises : whether there is any relation between the existence of repeated limits and that of double limit. In this connection, the following theorem is relevant :

Theorem : Let the double limit $\lim_{(x,y)\to(a,b)}$ f(x,y) exist and be equal to A ($\in \mathbb{R}$).

Let the limit $\lim_{x\to a} f(x,y)$ exist for each fixed value of y in the neighbourhood of *b* and like wise let the limit $\lim_{y\to b} f(x,y)$ exist for each fixed value of *x* in the neighbourhood of '*a*'.

Then $\lim_{x\to a}\lim_{y\to b} f(x,y) = A = \lim_{y\to b}\lim_{x\to a} f(x,y)$.

Proof : Let $\lim_{(x\to a)} f(x,y) = F(y)$ (by hypothesis, it exists)

Let ε > 0 be given let 0 < |y – b| < δ_0

Corresponding to ε, there exists $\delta_1 > 0$ such that

$|f(x,y)-F(y)|<\frac{\varepsilon}{2}$ ...(1) for all x satisfying

$0<|x-a|<\delta_1$. Also $0<|y-b|<\delta_0$. ...(1)

Also due to the existence of double limit, corresponding to above ε, there exists $\delta_2 > 0$ such that

$|f(x,y)-A|<\frac{\varepsilon}{2}$.....(1) for all x,y satisfying $0<|x-a|<\delta_2, 0<|y-b|<\delta_2$ ...(2)

Let $\eta = \min\{\delta_0, \delta_1, \delta_2\}$

So $|F(y)-A| \le |f(x,y)-F(y)|+|f(x,y)-A|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}$ whenever

$0<|y-b|<\eta$. So, $\lim_{y\to b} F(y)=A$.

Consequently $\lim_{y\to b}\lim_{x\to a} f(x,y)=A$

Similarly for the other part.

Illustration : Let $f(x,y)=\frac{(x-y)^2}{1-2xy+y^2}, (x,y)\neq(1,1)$

Examine the existence of $\lim_{(x,y)\to(1,1)} f(x,y)$

If possible, let the double limit exist & be equal to A

For $y \neq 1, \lim_{x\to 1} f(x,y)=\frac{(1-y)^2}{1-2y+y^2}=1 \Rightarrow \lim_{y\to 1}\lim_{x\to 1} f(x,y)=1$

For $x \neq 1, \lim_{y\to 1} f(x,y)=\frac{(1-x)^2}{2(1-x)}=\frac{1-x}{2} \Rightarrow \lim_{x\to 1}\lim_{y\to 1} f(x,y)=0$

So by above theorem ; the double limit does not exist.

6.4 Continuity at a point

Let f : $S \to \mathbb{R}$ where $S \subset \mathbb{R}^2$ and let $(a, b) \in$ S.

(a) If (a, b) be an accumulation point of S &

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

(b) Or If (a, b) be an isolated point of S,

then f is continuous at the point (a, b)

If for given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x,y) - f(a,b)| < \varepsilon \text{ wherever } \sqrt{[(x-a)^2 + (y-b)^2]} < \delta \text{ or}$$

$$(x,y) \in N((a,b)\delta) \cap S,$$

then f is continuous at (a, b)

Note : Let $g(x) = f(x, b)$. Then by above, $|g(x) - g(a)| < \varepsilon$ wherever $|x - a| \delta \Rightarrow g(x)$ is continuous at 'a'.

Similarly if $h(y) = f(a, y)$, then it is continuous at $y = b$.

But continuity of g(x) at a & that of $h(y)$ at b |⇒ continuity of $f(x, y)$ at (a, b)

Let f(x,y) = $\begin{cases} 0, \text{ if } xy \neq 0 \\ 1, \text{ if } xy = 0 \end{cases}$

Here $g(x) = f(x, 0) = 1$ for all $x \in \mathbb{R}$ & $h(y) = f(0, y) = 1$ for all $y \in \mathbb{R} \Rightarrow g(x), h(y)$ are continuous at $x = 0$, $y = 0$ respectively.

If possible, let $f(x,y)$ be continuous at (0, 0). Then for $\varepsilon = 1/2$ there exists $\delta > 0$ such that $|f(x,y) - f(0,0)| < \varepsilon$ whenever $(x,y) \in N((0,0)\delta) \cap \mathbb{R}$

$$\Rightarrow \left| f\left(\frac{\delta}{2}, \frac{\delta}{2}\right) - 1 \right| < \varepsilon \Rightarrow |1| < \varepsilon\left(= \frac{1}{2}\right) : \text{Absurd.}$$

Thus f is not continuous at (0, 0),

Examples : (1) $f(x,y) = \begin{cases} xy\log(x^2+y^2), x^2+y^2 \neq 0 \\ 0, \; x^2+y^2 = 0 \end{cases}$

For $0 < x^2 + y^2 < 1$, $\log(x^2 + y^2) < 0$ & so we have

$$|f(x,y) - f(0,0)| = -|xy|\log(x^2 + y^2) \leq -\frac{1}{2}(x^2 + y^2)\log(x^2 + y^2) \quad [AM \geq GM]$$

If we have $x^2 + y^2 = t$, by L Hospital's rule, $\lim_{t \to 0^+} t \log t = 0$

So given $\varepsilon > 0$, there exists $\delta > 0$ such that

$|t \log t - 0| < 2\varepsilon$ whenever $0 < t < \delta$

$\Rightarrow |(x^2 + y^2)\log(x^2 + y^2)| < 2\varepsilon$ whenever $0 < t < \delta$

For $\eta = \min\{1, \delta\}$, we have for $0 < x^2 + y^2 < \eta$,

$|f(x,y) - f(0,0)| < \varepsilon \Rightarrow f$ is continuous at (0, 0).

(2) Let $f(x,y) = \begin{cases} 0, \text{ if } (x,y) = (2y, y) \\ \exp\{-|x-2y|/(x^2 - 4xy + 4y^2)\}, (x,y) \neq (2y, y) \end{cases}$

we note that $\dfrac{|x-2y|}{x^2 - 4xy + 4y^2} = \dfrac{1}{|x-2y|}$

Let $0 < \varepsilon < 1$ and $\varepsilon_1 = \dfrac{-1}{\log \varepsilon}$

$$|x - 2y| \leq |x| + 2|y| < \frac{\varepsilon_1}{4} + \frac{2\varepsilon_1}{4} \text{ whenever } |x - 0| < \frac{\varepsilon_1}{4}, |y - 0| < \frac{\varepsilon_1}{4} \Rightarrow |x - 2y| < \varepsilon_1$$

$$\Rightarrow \left\{ |x-2y|/(x^2 - 4xy + 4y^2) \right\} > \frac{1}{\varepsilon_1} \Rightarrow \frac{-|x-2y|}{(x^2 - 4xy + 4y^2)} < -\frac{1}{\varepsilon_1} = \log \varepsilon$$

$$\Rightarrow \exp\left\{ \frac{-|x-2y|}{x^2 - 4xy + 4y^2} \right\} < \varepsilon$$

$\Rightarrow |f(x,y) - f(0,0)| < \varepsilon$ whenever $|x - 0| < \delta, |y - 0| < \delta, \delta \equiv \delta(\varepsilon)$.

Thus f is continuous at (0, 0).

(3) Let f and g be two functions of one variable which are continuous on $[a-\delta_1, a+\delta_1]$ & $[b-\delta_2, b+\delta_2]$ respectively, $\delta_1 > 0, \delta_2 > 0$.

If $h(x,y) = \max\{f(x), g(y)\}$, then h is continuous on

$\left[a-\delta_1, a+\delta_1; b-\delta_2, b+\delta_2\right]$

Here, $h(x,y) = \frac{1}{2}\left[f(x)+g(y)+|f(x)-g(y)|\right]$

Let (x', y') $\in [a-\delta_1, a+\delta_1; b-\delta_2, b+\delta_2]$

Let $\varepsilon > 0$ be any number

As f is continuous at x', so corresponding to above ε, there exists δ,

$0 < \delta < \delta_1$, such that $|f(x) - f(x')| < \frac{\varepsilon}{2}$ whenever $|x - x'| < \delta$. As g is continuous at y', corresponding to above ε there exists δ′,

$0 < \delta' < \delta_2$, such that $|g(y) - g(y')| < \frac{\varepsilon}{2}$ whenever $|y - y'| < \delta'$.

Let $\eta = \min\{\delta, \delta'\}$

So $|f(x) - f(x')| < \frac{\varepsilon}{2}$ whenever $|x - x'| < \eta$ & $|g(y) - g(y')| < \frac{\varepsilon}{2}$ whenever

$|y - y'| < \eta$

Consequently $|\{f(x) \pm g(y)\} - \{f(x') \pm g(y')\}|$

$\leq |f(x) - f(x')| + |g(y) - g(y')| < \varepsilon$ whenever $|x - x'| < \eta, |y - y'| < \eta$

Therefore $f(x) + g(y)$ and $f(x) - g(y)$ are continuous at (x', y'), $|f(x) - g(y)|$ continuous at (x', y'). So their sum, difference and scalar multiple are continuous. Hence $h(x,y)$ is continuous at (x', y').

(4) Let $f(x,y) = \begin{cases} \dfrac{x^\alpha y^\beta}{x^2 + xy + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Examine for the continuity of f(x, y) at (0, 0).

Let $(x, y) \to (0, 0)$ along the line $y = mx$.

$$\frac{x^\alpha y^\beta}{x^2+xy+y^2}=\frac{m^\beta}{1+m+m^2}.x^{\alpha+\beta-2}$$

If $\alpha+\beta=2$ we get $\dfrac{m^\beta}{1+m+m^2}$ & if $\alpha+\beta<2$, the limit does not exist

So $\alpha+\beta \nleq 2$

Let us **consider the case** $\alpha+\beta>2$

We put $x=r\cos\theta, y=r\sin\theta$, Then,

$$\frac{x^\alpha y^\beta}{x^2+xy+y^2}=r^{\alpha+\beta-2}\frac{(\cos\theta)^\alpha(\sin\theta)^\beta}{1+\sin\theta\cos\theta}$$

For any $\theta, \frac{1}{2}\le 1+\sin\theta\cos\theta\le 2.$

Then, $|f(x,y)|=\left|r^{\alpha+\beta-2}\dfrac{(\cos\theta)^\alpha(\sin\theta)^\beta}{1+\sin\theta\cos\theta}\right|\le 2r^{\alpha+\beta-2}\quad 2r^{\alpha+\beta+2}\to 0$ as $r\to 0$

provided $\alpha + \beta > 2$.

Therefore, when $\alpha + \beta > 2$, $|f(x,y)-f(0,0)|<\varepsilon$ whenever

$|x-0|<\delta, |y-0|<\delta, \delta\equiv\delta(\varepsilon)\Rightarrow f$ is continuous at (0, 0) only when $\alpha + \beta > 2$.

(5) Let f(x,y) = $\begin{cases}(ax+by)\sin\dfrac{x}{y}, y\neq 0\ (a,b\in\mathbb{R})\\ 0, y=0\end{cases}$

Test for continuity of $f(x, y)$ at (0, 0)

Let $\varepsilon > 0$ be any number

$$|f(x,y)-f(0,0)|=\left|(ax+by)\sin\frac{x}{y}-0\right|\le|ax+by|\le|a||x|+|b||y|$$

$< \dfrac{|a|.\varepsilon}{2(|a|+1)}+\dfrac{|b|.\varepsilon}{2(|b|+1)}$ whenever

$$|x-a|<\delta_1=\frac{\varepsilon}{2(|a|+1)},\ |y-b|<\delta_2=\frac{\varepsilon}{2(|b|+1)}$$

If $\delta=\min\{\delta_1,\delta_2\}$, we have

$|f(x,y)-f(0,0)|<\varepsilon$ whenever $|x-0|<\delta, |y-0|<\delta\,;\,\delta\equiv\delta(\varepsilon)$

So f is continuous at (0, 0).

(6) Let f(x, y) $=\begin{cases}\dfrac{x^6-2y^4}{x^2+y^2}, x^2+y^2\neq 0\\ 0,\ x^2+y^2=0\end{cases}$

Test for the continuity of f(x, y) at (0, 0).

Put $x=r\cos\theta, y=r\sin\theta$

$\dfrac{x^6-2y^4}{x^2+y^2}=r^4\cos^6\theta-2r^2\sin^4\theta.$ Let $\varepsilon>0$ be any number

$|f(x,y)-f(0,0)|\leq r^4+2r^2<\varepsilon$ whenever $x^2+y^2<\delta\ \&\ \delta=\sqrt[4]{\dfrac{\varepsilon}{4}}$

$\Rightarrow f$ is continuous at (0, 0).

(7) $f(x,y)=\begin{cases}\dfrac{2x^3-y^3}{x^2+y^2}, x^2+y^2\neq 0\\ 0,\ x^2+y^2=0\end{cases}$

Put x = $r\cos\theta, y=r\sin\theta,$ Then $\dfrac{2x^3-y^3}{x^2+y^2}=2r\cos^3\theta-r\sin^3\theta.$

$|f(x,y)-f(0,0)|\leq 2r+r<\varepsilon$ whenever $r<\delta=\dfrac{\varepsilon}{3}(\varepsilon>0$ is arbitrary)

$\Rightarrow f(x,y)$ is continuous at (0,0).

6.5 Neighbourhood Properties

(1) If $\lim\limits_{(x,y)\to(p,q)} f(x,y)=\lambda(\in\mathbb{R})$, there exists a deleted neighbourhood of (p, q) in which f is bounded.

We note that there exists $\delta > 0$ such that

$$|f(x,y)-\lambda| < \frac{1}{2} \text{ whenever } (x, y) \in N'((p,q),\delta) \cap D_f$$

$$\Rightarrow |f(x,y)| \le |f(x,y)-\lambda| + |\lambda| < \frac{1}{2} + |\lambda| \text{ for all } (x,y) \in N'((p,q),\delta) \cap D_f$$

$$\Rightarrow f(x,y) \text{ is bounded in } N'((p,q),\delta) \cap D_f$$

(2) If $\lim_{(x,y)\to(p,q)} f(x,y) = \lambda,\ \lambda \in \mathbb{R} - \{0\}$, there exists a deleted neighbourhood of (p, q) in which $f(x, y)$ does not vanish.

There exists $\eta > 0$ such that

$$|f(x,y)-\lambda| < \frac{|\lambda|}{2} \text{ whenever } (x,y) \in N'((p,q),\eta) \cap D_f$$

$$\Rightarrow \frac{1}{2}|\lambda| < |f(x,y)| < \frac{|\lambda|}{2} + |f(x,y)|,\ (x,y) \in N'((p,q),\eta) \cap D_f$$

LHS of the last inequality implies that f(x,y) does not vanish in N' $((p,q),\eta) \cap D_f$

(3) Let f be defined and continuous in S ($\subset \mathbb{R}^2$). If at two points $M'(x', y')$ and M'' (x", y") of S, $M'M''$ lies entirely in S, the function takes values of distinct signs, say $f(x', y') < 0, f(x'', y'') > 0$, then there exists a point M_0 (x_0, y_0) in the domain at which $f(x_0, y_0) = 0$.

Let $x = x' + t(x'' - x'), y = y' + t(y'' - y')$ where $0 \le t \le 1$.

$f(x,y) = f(x' + t(x'' - x'), y' + t(y'' - y')) = F(t)$.

By hypothesis $F(t)$ is continuous in [0, 1]. $F(0)\,F(1) = f(x', y')\,f(x'', y'')$ i.e., $F(0)\,F(1) < 0$. By Bolzano's theorem on continuous function for a function of one variable, there exists a point $t_0 \in (0,1)$ for which $F(t_0) = 0$ => there is a point (x_0, y_0) such that $f(x_0, y_0) = 0$ where

$x_0 = x' + t_0\,(x'' - x'),\ y_0 = y' + t_0\,(y'' - y')$

Some important results :

We state the following results without proof

Results (1) : If functions $\phi_i(P)\ (i = 1,2)$ are continuous at the point $P'\ (t_1', t_2')$ in $S(\subset \mathbb{R}^2)$ and the functions $f(M)$, $M(x_1, x_2)$, be continuous at the corresponding point $M'(x_1', x_2')$ where $x_1' = \varphi_1(t_1', t_2')$ $x_2' = \varphi_2(t_1', t_2')$ then the composite function

$u = f(\varphi_1(t_1, t_2), \varphi_2(t_1, t_2)) = f(\varphi_1(p), \varphi_2(p))$ is continuous at p'.

Result (2) : If f(x, y) is defined & continuous in a bounded closed domain $S(\subset \mathbb{R}^2)$, then f is bounded above and below in S and f attains its bounds in S.

Example : Let $f(x, y)$ be defined in the square $S = \{(x,y) | |x| \le 1, |y| \le 1\}$

$$f(x,y) = \begin{cases} \dfrac{xy}{x^4 + y^4}, & x^4 + y^4 \neq 0 \\ 0, & x^4 + y^4 = 0 \end{cases}$$

Examine for continuity and boundedness of f on S.

It is evident that f is continuous in x for every y and f is continuous in y for every x. To discuss double limit, let $(x, y) \to (0, 0)$ along $y = mx^3$.

Then $f(x,y) = \dfrac{mx^4}{x^4 + m^4x^{12}} = \dfrac{m}{1 + m^4x^8} \to m$ as $x \to 0$

So double limit $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist & f is not continuous at (0, 0).

If possible, let f be bounded. Then there would exist M > 0 such that $|f(x,y)| < M$ for all (x, y) $\in S$.

For $x = y = \dfrac{1}{2\sqrt{M}}$, $f\left(\dfrac{1}{2\sqrt{M}}, \dfrac{1}{2\sqrt{M}}\right) = \dfrac{1}{4M} / \left(\dfrac{1}{16M^2} + \dfrac{1}{16M^2}\right) = 2M > M$

This indicates $f(x, y)$ is not bounded on S.

6.6 Summary

In this chapter we have introduced the concept of limit and continuity for function of two variable as a generalization of one variable. We also have examined condition for the existence of double limit. We have studied repeated limits and continuity at a point with some examples. We have further developed the neighbourhood properties.

Unit-7 □ Partial Differentiation

Structure

7.0 Objectives

This unit gives

- The concept of partial differentiation
- The derivative of a function along some direction
- The concept of total differential
- Chain rule and some application

7.1 Introduction

Suppose that f is a function of more than one variable. For instance, $z = f(x,y) = x^2 + xy + y^2$. The graph of this function defines a surface in Euclidean space. To every point on this surface, there are an infinite number of tangent lines. Partial differentiation is the act of choosing one of these lines and finding its slope. Usually, the lines of most interest are those that are parallel to the xy-plane, and those that are parallel to the yz-plane.

7.2 Partial Differentiation

U21 Let $f : S \to \mathbb{R}$ where S ($\subset \mathbb{R}^2$) be an open set

Let (a, b) be an interior point of S.

(i) If $\lim\limits_{x\to a}\dfrac{f(x,b)-f(a,b)}{x-a}$ exists as a finite, definite number, it is called first order partial derivative of f *w.r.t.* x at (a, b). This is denoted by $f_x(a, b)$ or $\dfrac{\partial f}{\partial x}|(a,b)$

We can also write $f_x(x,y) = \lim\limits_{h\to 0}\dfrac{f(x+h,y)-f(x,y)}{h}$, if limit exists

(ii) If $\lim\limits_{y\to b}\dfrac{f(a,y)-f(a,b)}{y-b}$ exists as a finite, definite number, it is called first order partial derivative of f w.r.t. y at (a, b). This is denoted by $f_y(a,b)$ or $\dfrac{\partial f}{\partial y}(a,b)$

$f_y(x,y) = \lim\limits_{k\to 0}\dfrac{f(x,y+k)-f(x,y)}{k}$, if limit exists.

Geometrical significance :

Let $Z = f(x, y)$ represent the surface S. Let f have first order partial derivatives w.r.t. x and y at each point of its domain. If (a, b) is such a point, let $c = f(a, b)$. So (a, b, c) is a point on the surface S.

To find f_x at (a, b), we hold y = constant = b, The equation $Z = f(x, y)$, $y = b$ defines the curve in which the plane $y = b$ cuts the surface S. $f_x(a, b)$ is the slope of the curve (1) relative to the x-direction at the point (a,b,c).

Similarly, the plane $x = a$ cuts the surface S in a curve $Z = f(x,y)$, $x = a$ (2) whose slope relative to the y-direction at $(a, b$, c) is f_y (a, b).

Examples : (1) Let $f(x,y)=\begin{cases} x\sin\frac{1}{x}+y\sin\frac{1}{y}, & xy\neq 0 \\ x\sin\frac{1}{x}, & x\neq 0, y=0 \\ y\sin\frac{1}{y}, & x=0, y\neq 0 \\ 0, & x=0=y \end{cases}$

As $\lim_{t\to 0}\sin\frac{1}{t}$ does not exist, so neither f_x nor f_y exists at (0, 0),

Note that f is continuous at (0, 0). For $\varepsilon>0$ any number

$$|f(x,y)-f(0,0)|=\left|x\sin\frac{1}{x}+y\sin\frac{1}{y}-0\right|\leq|x|+|y|<\varepsilon \text{ whenever}$$

$$|x-0|<\delta, |y-0|<\delta \;\&\; \delta=\varepsilon/2$$

(2) Let $f(x,y)=(|x+y|+x+y)^k,\ (x,y)\in\mathbb{R}^2$

Examine for the existence of f_x and f_y at (0, 0)

We first note that f can be defined only when $k>0$, $(x, y)\in\mathbb{R}^2$

$\lim_{h\to 0+}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0+}2^k h^{k-1}$ exist only when $k>1$.

If $0<k<1$, the limit does not exist.

If $k>1$, the limit is zero & if $k=1$, the limit is 2.

$$\lim_{h\to 0-}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0-}\frac{0-0}{h}=0$$

So, for existence of f_x (0, 0), we must have $Rf_x(0,0)=Lf_x(0,0)=0$ & that is possible only when $k>1$.

Similarly, for existence of f_y at (0, 0), **we must have $k>1$.**

(3) Let $f(x,y)=\begin{cases}\cos\left(\frac{\pi}{2}.\frac{x^2-y^2}{x^2+y^2}\right), x^2+y^2>0\\ 0, x^2+y^2=0\end{cases}$

Examine for the continuity of f at (0, 0) & existence of f_x, f_y at (0, 0).

To examine for continuity of f at (0, 0), let us opt for sequential approach.

Note that $\left\{\frac{1}{n},\frac{1}{n}\right\}_n \to (0,0)$

$$f\left(\frac{1}{n},\frac{1}{n}\right)=\cos\left(\frac{\pi}{2}.0\right)=1 \text{ \& so, } \lim_{n\to\infty} f\left(\frac{1}{n},\frac{1}{n}\right)\neq f(0,0)$$

$\Rightarrow f$ is not continuous at (0, 0).

$$\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0 \Rightarrow f_x(0,0)=0$$

$$\lim_{K\to 0}\frac{f(0,0+k)-f(0,0)}{k}=\lim_{k\to 0}\frac{0-0}{k}=0 \Rightarrow f_y(0,0)=0$$

So note that though first order *p. d.*' s exist at (0, 0), f is not continuous at (0, 0). This nature of $f(x, y)$ is a major departure / change from the property of function of one variable. Here existence of first order p.d.'s at a point does not ensure the continuity of $f(x, y)$ at that point.

Sufficient condition for continuity at a point :

A sufficient condition for the continuity of $f(x, y)$ at (a, b) (as stated above) is that one of the first order p.d's exist at that point and the other p.d. exists and is bounded in the neighbourhood of that point.

Proof : Let $f_x(a, b)$ exist and f_y be bounded in a neighbourhood of (a, b), say $N((a, b), \delta)$.

We choose h, k so that $(a + h, b + k), (a + h, b) \in N((a, b), \delta)$

$f(a + h, b + k) - f(a, b) = f(a + h, b+k) - f(a+h, b) + f(a+h, b) - f(a,b)$ (1)

As $f_x(a, b)$ exists, so $\lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}=f_x(a,b)$

$$\Rightarrow if\ \frac{f(a+h,b)-f(a,b)}{h}-f_x(a,b)=\eta(h), \text{ then } \eta\to 0 \text{ as } h\to 0$$

$\Rightarrow f(a+h,b)-f(a,b)=h\{f_x(a,b)+\eta\}$ where $\eta \to 0$ as $h \to 0$

Let $f(a+h,y)=g(y)$ so $f(a+h,b+k)-f(a+h,b)=g(b+k)-g(b)$.

As f_y exists in $N((a,b),\delta)$, g is derivable in $[b,\ b+k]$ or in $[b+k,\ b]$.

By Lagrange's mean value theorem, there exists at least one

$\theta \in (0,1)$ such that $g(b+k) - g(b) = kf_y\,(a+h,\ b+\theta k)$

$\Rightarrow f(a+h,b+k)-f(a+h,b)=kf_y(a+h,b+\theta k)$

Recalling (1), $f(a+h,b+k)-f(a,b)=h\{f_x(a,b)+\eta\}+kf_y(a+h,b+\theta k)$

where $\eta \to 0$ as $h \to 0$

Hence $\lim\limits_{(h,k)\to(0,0)} \{f(a+h,b+k)-f(a,b)\}=0$ (f_y being bounded in

$N(a,b,\delta)\bigcap D_f$

$\Rightarrow \lim\limits_{(h,k)\to(0,0)} f(a+h,b+k)=f(a,b)$ & f is continuous at $(a,\ b)$

Remark, Another set of sufficient condition for continuity of $f(x,\ y)$ at $(a,\ b)$ is that f_x exists & is bounded in neighbourhood of $(a,\ b)$ & f_y exists at $(a,\ b)$.

Illustration :

Let $f:\mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y)=\begin{cases} \dfrac{y^3}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

Here $\lim\limits_{h\to 0}\dfrac{f(0+h,0)-f(0,0)}{h}=\lim\limits_{h\to 0}\dfrac{0-0}{h}=0 \Rightarrow f_x(0,0)=0$

Also $\lim\limits_{K\to 0}\dfrac{f(0,0+k)-f(0,0)}{k}=\lim\limits_{k\to 0}\dfrac{k-0}{k}=1 \Rightarrow f_y(0,0)=1$

Here $f_x(x,y)=\begin{cases} \dfrac{-2xy^3}{(x^2+y^2)^2}, & x^2+y^2 \neq 0 \\ 0 & x^2+y^2=0 \end{cases}$ &

$$f_y(x,y)=\begin{cases}\dfrac{(3x^2+y^2)y^2}{(x^2+y^2)^2}, x^2+y^2\neq 0\\ 1, x^2+y^2=0\end{cases}$$

We note that $\dfrac{x^2+y^2}{2}\geq |xy|$ & $\dfrac{y^2}{x^2+y^2}\leq 1, (x^2+y^2\neq 0)$

$\Rightarrow \left|\dfrac{2xy^3}{(x^2+y^2)^2}\right|\leq 1$ for all (x, y) in deleted neighbourhood of (0, 0).

Considering $f_x(0, 0)$, we say that f_x is bounded in neighbourhood of (0, 0). Hence f is continuous at (0, 0).

Examples : Let $u(x,y)=(1-2xy+y^2)^{-1/2}$, show that

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\}=0$$

Here $u_x=\dfrac{y}{(1-2xy+y^2)^{3/2}}=yu^3, u_y=-\dfrac{1}{2}.\dfrac{2(y-x)}{(1-2xy+y^2)^{3/2}}=(x-y)u^3$

$$\frac{\partial}{\partial x}(u_x)=y.\,3u^2.u_x=3y^2u^5$$

$$\frac{\partial}{\partial y}(u_y)=-u^3+(x-y)^2.3u^2.u^3=-u^3+3u^5(x-y)^2$$

Given expression = $-2x.u_x+2y.u_y+(1-x^2)3y^2u^5+y^2\{-u^3+3u^5(x-y)^2\}$

$=-3y^2u^3+3y^2u^5(1-2xy+y^2)=-3y^2u^3+3y^2u^3=0.$

(2) If $\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}=1$ where $u\equiv u(x,y)$. Show that

$$(u_x)^2+(u_y)^2=2(xu_x+yu_y).$$

Differentiating given relation,

$$\frac{2x}{a^2+u}+x^2\left\{\frac{-1}{(a^2+u)^2}\right\}.u_x+y^2.\left\{\frac{-1}{(b^2+u)^2}.u_x\right\}=0$$

$$\Rightarrow u_x=\frac{2x/(a^2+u)}{\dfrac{x^2}{(a^2+u)^2}+\dfrac{y^2}{(b^2+u)^2}}, \text{ Similarly } u_y=\frac{2y/(b^2+u)}{\dfrac{x^2}{(a^2+u)^2}+\dfrac{y^2}{(b^2+u)^2}}.$$

$$\text{So, } xu_x+yu_y=\frac{2\left\{\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}\right\}}{\dfrac{x^2}{(a^2+u)^2}+\dfrac{y^2}{(b^2+u)^2}}=\frac{2}{\dfrac{x^2}{(a^2+u)^2}+\dfrac{y^2}{(b^2+u)^2}}.$$

$$\text{Also } (u_x)^2+(u_y)^2=\frac{\dfrac{4x^2}{(a^2+u)^2}+\dfrac{4y^2}{(b^2+u)^2}}{\left\{\dfrac{x^2}{(a^2+u)^2}+\dfrac{y^2}{(b^2+u)^2}\right\}^2}=\frac{4}{\dfrac{x^2}{(a^2+u)^2}+\dfrac{y^2}{(b^2+u)^2}}$$

Hence $2(xu_x+yu_x)=(u_x)^2+(u_y)^2$

Sufficient condition for continuity in a region :

If $f(x,y)$ has partial derivatives f_x, f_y everywhere in a region D & their derivatives every where in D satisfy the inequalities

$$|f_x(x,y)|<M, |f_y(x,y)|<M$$

where M is independent of x and y, then f(x,y) is continuous everywhere in D.

Proof : Let (a, b), $(a + h, b + k) \in D$.

We can write $f(a + h, b + k) - f(a, b) = f(a + h, b + k)$

$$- f(a + h, b) + f(a + h, b) - f(a, b)$$

By applying LMV theorem, there exists $\theta_1, \theta_2 \in (0,1)$ such that

$$f(a+h,b+k)-f(a+h,b)=kf_y(a+h,b+\theta_1 k)$$

$$f(a+h,b)-f(a,b)=hf_x(a+\theta_2 h,b)$$

So $\left|f(a+h,b+k)-f(a,b)\right| \le |k||f_y(a+h,b+\theta_1 k|+|h||f_x(a+\theta_2 h,b)|$

$$<|k|M+|h|M<M.\frac{\varepsilon}{2M}+M.\frac{\varepsilon}{2M} \text{ where}$$

$$|h|<\delta, |k|<\delta, \delta=\frac{\varepsilon}{2M} (\varepsilon>0 \text{ being arbitrary})$$

$\Rightarrow f$ is continuous at any point (a, b) of D.

Hence the result follows.

7.3 Directional Derivative

In the definition of $f_x(x_0, y_0)$, the point $(x_0+\Delta x, y_0)$ approaches the point (x_0, y_0) along the line $y = y_0$ and in the definition of f_y (x_0, y_0), the point $(x_0, y_0 + \Delta y)$ approaches the point (x_0, y_0) along the line $x = x_0$. Let us consider a generalisation of these concepts by replacing the above two special lines by an arbitrary line through (x_0, y_0).

A direction ξ_α is used to designate as the direction of any directed line which makes an angle α with the positive side of x-axis measured in the anti-clockwise direction.

We define $\frac{\partial f}{\partial \xi_\alpha}\Big|_{(a,b)} = \lim\limits_{\Delta s\to 0}\frac{f(a+\Delta s\cos\alpha, b+\Delta s\sin\alpha)-f(a,b)}{\Delta s}$, provided the limit exists.

This means the rate of change of f at the point (x, y) w.r.t. distance as we approach (x, y) along the ray that forms an angle α with the positive side of x-axis.

Result : Let $f(x, y)$ have continuous first order partial derivatives. Then

$$\frac{\partial f}{\partial \xi_\alpha}=f_x(x,y).\cos\alpha+f_y(x,y).\sin\alpha$$

i.e. the directional derivative is a linear combination of f_x and f_y.

Proof : $f(a+\Delta s\cos\alpha, b+\Delta s\sin\alpha)-f(a,b)$

$$=\{f(a+\Delta s\cos\alpha,b)-f(a,b)\}+\{f(a+\Delta s\cos\alpha,b+\Delta s\sin\alpha)-f(a+\Delta s\cos\alpha,b)\}$$

Let $f(x,b) = t(x)$. Then

$$f(a+\Delta s\cos\alpha,b)-f(a,b)=t(a+\Delta s\cos\alpha)-t(a)$$

By the conditions imposed on f, t is derivable in $[a, a+\Delta s\cos\alpha]$.

By LMV theorem, there exists $\theta_1 \in (0,1)$ such that

$$t(a+\Delta s\cos\alpha)-t(a)=\Delta s\cos\alpha . t'(a+\theta_1.\Delta s\cos\alpha)$$

$$=\Delta s\cos\alpha . f_1(a+\theta_1.\Delta s\cos\alpha, b)$$

and taking $f(a+\Delta s\cos\alpha, y)=g(y)$ & arguing in a similar way, we get

$$f(a+\Delta s\cos\alpha, b+\Delta s\sin\alpha)-f(a+\Delta s\cos\alpha, b)$$

$$=\Delta s\sin\alpha . f_2(a+\Delta s\cos\alpha, b+\theta_2\Delta s\sin\alpha) \text{ for some } \theta_2 \in (0,1)$$

(f_i denotes the p.d. of f w.r.t. ith component).

Hence $\dfrac{f(a+\Delta s\cos\alpha, b+\Delta s\sin\alpha)-f(a,b)}{\Delta s}=$

$$f_1(a+\theta_1.\Delta s\cos\alpha, b).\cos\alpha+f_2(a+\Delta s\cos\alpha, b+\theta_2\Delta s\sin\alpha).\sin\alpha$$

Taking $\Delta s \to 0$ & using the continuity of f_x and f_y, we get

$$\frac{\partial f}{\partial \xi_\alpha}=f_x(a,b).\cos\alpha+f_y(a,b).\sin\alpha$$

Illustration : Let $f(x,y)=\sqrt{(2x^2+y^2)},\ a=1=b; \alpha=60^0$

$$f_x=\frac{2x}{\sqrt{(2x^2+y^2)}}, f_y=\frac{y}{\sqrt{(2x^2+y^2)}}$$

$$\frac{\partial f}{\partial \xi_\alpha}=f_x(1,1)\cos 60^0+f_y(1,1)\sin 60^o=\frac{2}{\sqrt{3}}.\frac{1}{2}+\frac{1}{\sqrt{3}}.\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{3}+\frac{1}{2}$$

Remarks : (1) The directional derivative can also be defined as follows :

Let $f : S \to \mathbb{R}$ where $S \subset \mathbb{R}^2$ and c be an interior point of S. Let the line-segment joining c and $c + hu$ lie in $B(c, r)$ ($\subset S$). Here $h \in \mathbb{R}$ and u describes the direction of line segment.

Then the directional derivative of f at c in the direction of the unit vector **u**, denoted by $f'(c, u)$ is given by

$$f'(c,u)=\lim_{h\to 0}\frac{f(c+hu)-f(c)}{h} \text{ if limit exists.}$$

If $u = \langle a,b \rangle$ & $c \equiv (x_0, y_0)$, we can write

$$\lim_{h\to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h},$$ if limit exists.

(2) The directional derivative can also be defined as $(f_x, f_y) \cdot (p, q)$

where (p, q) denotes the unit vector along the given direction $\langle a, b \rangle$ and $\cdot$ denotes the usual inner product (dot product).

Note : Existence of p.d's at a point does not ensure the existence of directional derivative in any other direction.

For example $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} x+y, \text{if } xy = 0 \\ 1, \text{otherwise} \end{cases}$$

$$\lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1 \Rightarrow f_x(0,0) = 1$$

$$\lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{k-0}{k} = 1 \Rightarrow f_y(0,0) = 1$$

$$\lim_{t\to 0} \frac{f(ta,tb) - f(0,0)}{t} = \lim_{t\to 0} \frac{1}{t}$$ does not exist.

7.4 Differentiability at a point

Let $f : S \to \mathbb{R}$, S be open subset of $\mathbb{R}^2$ and $(a, b) \in$ S.

We say that f is differentiable at (a, b) if

$f(a+h,b+k) - f(a,b) = Ah + Bk + h\phi(h,k) + k\psi(h,k)$ (1) where A, B are independent of h, k and $\varphi \to 0, \psi \to 0$ as $h \to 0$, $k \to 0$.

For $k = 0$, $h \neq 0$

$$\frac{f(a+h,b) - f(a,b)}{h} = A + \phi$$

As $h \to 0, \phi \to 0,$ & so as $h \to 0, \frac{f(a+h,b) - f(a,b)}{h}$ tends to A.

$\Rightarrow f_x(a,b)$ exists = A.

Similarly taking h = 0, $k \neq 0$, we get

$$\lim_{K\to 0}\frac{f(a,b+k)-f(a,b)}{k} \text{exists \& } = B.$$

$\Rightarrow f_y(a,b)$ exists & = B.

Also in (1), taking $(h, k) \to (0, 0)$ we get

$$\lim_{(h,k)\to(0,0)} f(a+h,b+k)=f(a,b)$$

So differentiability at a point implies (i) existence of first order partial derivatives at that point (ii) continuity of f at that point.

An important note :

In case of function of one variable,

differentiability at a point < = > existence of derivative at that point .

But in case of functions $\mathbb{R}^2 \to \mathbb{R}$, differentiability at a point $\Rightarrow$ existence of first order partial derivatives at that point (as shown above) but the converse is not true.

Examples :

$$\textbf{(1)}\ \ f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{(x^2+y^2)}} x^2+y^2\neq 0\\ 0,\ x^2+y^2=0\end{cases}$$

$$\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0,\lim_{k\to 0}\frac{f(0,0+k)-f(0,0)}{k}=\lim_{k\to 0}\frac{0-0}{k}=0$$

$\Rightarrow f_x(0,0)=0=f_y(0,0)$.

In order to be differentiable at (0, 0), we must have

$f(0+h,0+k)-f(0,0)=hf_x(0,0)+kf_y(0,0)+h\phi+k\psi$

where $\phi \to 0, \psi \to 0$ as $h \to 0, k \to 0$

$$\Rightarrow \frac{hk}{\sqrt{(h^2+k^2)}}-0=h.0+k0+h\varphi+k\psi$$

In particular if h = k, $\frac{h}{\sqrt{2}} = h(\varphi + \psi) \Rightarrow \frac{1}{\sqrt{2}} = \varphi + \psi$.

As $h \to 0, \varphi + \psi \to 0$ but LHS $\neq 0$

So f is not differentiable at (0, 0).

(2) Let $f(x,y) = \begin{cases} x, if \mid y \mid < \mid x \mid \\ -y, if \mid y \mid \geq \mid x \mid \end{cases}$

$\lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{h-0}{h} = 1$ (Here $h \to 0$, $k = 0$ & so $|h|>|k|$)

$\lim_{K \to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{K \to 0} \frac{-k-0}{k} = -1$ (Here $k \to 0$, $h = 0$ & so $|k|>|h|$)

In order to be differentiable at (0, 0), we must have

$f(0+h, 0+k) - f(0,0) = h \cdot 1 + k(-1) + h\varphi + k\psi$

where $\varphi \to 0, \psi \to 0 \; as \; h \to 0, k \to 0$.

In particular, if $k = -h$, we get

$-k = -2k - k\varphi + k\psi \Rightarrow 1 = \psi - \varphi$.

we have $RHS \to 0, LHS \neq 0$ as $k \to 0$

So f is not differentiable at (0,0).

(3) Let $f(x,y) = \begin{cases} \frac{x^3 - y^3}{x^2 + y^2}, x^2 + y^2 \neq 0 \\ 0, \; x^2 + y^2 = 0 \end{cases}$

$$\lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{h-0}{h} = 1 \Rightarrow f_x(0,0) = 1$$

$\lim_{K \to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k \to 0} \frac{-k-0}{k} = -1 \Rightarrow f_y(0,0) = -1$

In order to be differentiable at (0,0), we must have

$f(0+h,0+k) - f(0,0) = hf_x(0,0) + kf_y(0,0) + h\varphi + k\psi$

where $\varphi \to 0, \psi \to 0$ as $h \to 0, k \to 0$

$$\Rightarrow \frac{h^3-k^3}{h^2+k^2} = h.1 + k(-1) + h\varphi + k\psi$$

In particular, if $k = 5h$,

$$\frac{h^3-125h^3}{h^2+25h^2} = h - 5h + h\varphi + 5h\psi \Rightarrow \frac{-124}{26} = -4 + \varphi + 5\psi$$

As $h \to 0, RHS \to -4$ but $LHS = \dfrac{-124}{26}$

So f is not differentiable at (0, 0).

Sufficient condition for differentiability at a point :

Let $f : S \to \mathbb{R}$ where $S \subset \mathbb{R}^2$ and let $(a, b) \in S$.

Let (i) f_x exist at (a, b) (ii) f_y be continuous at (a, b).

Then f is differentiable at (a, b).

Proof : By hypothesis, there exists a neighbourhood of (a, b), say $[a-\delta, a+\delta\,; b-\delta, b+\delta]\,(\delta > 0)$ in which both f, f_y are defined.

we take $h, k, h^2 + k^2 \neq 0$, so that

$$(a+h, b+k), (a+h, b) \in N((a,b),\delta).$$

$$f(a+h, b+k) - f(a,b)$$

$$= \{f(a+h,b+k) - f(a+h,b)\} + \{f(a+h,b) - f(a,b)\}$$

As $f_x(a,b)$ exists, $f_x(a,b) = \lim\limits_{h\to 0} \dfrac{f(a+h,b)-f(a,b)}{h}$

$$\Rightarrow if\ \frac{f(a+h,b)-f(a,b)}{h} - f_x(a,b) = \varepsilon, \text{ then } \varepsilon \to 0, \text{as } h \to 0$$

$$\Rightarrow f(a+h,b) - f(a,b) = hf_x(a,b) + \varepsilon h \text{ where } \varepsilon \to 0\ as\ h \to 0$$

Next let $f(a+h, y) = g(y)\ \&\ so\ f(a+h,b+k) - f(a+h,b) = g(b+k) - g(b)$

We note that existence of $f_y in\ N((a,b),\delta) \Rightarrow$ existence of

$g'(y)\ in\ [b, b+k]\ or\ in\ [b+k, b]$

By LMV theorem, there exists $\theta \in (0,1)$ such that

$$g(b+k)-g(b)=kg'(b+\theta k)=kf_y(a+h,b+\theta k)$$

Consequently $f(a+h,b+k)-f(a,b)=hf_x(a,b)+kf_y(a+h,b+\theta k)+\varepsilon h.$

By hypothesis, f_y is continuous at (a, b), So

$$\lim_{\substack{h\to 0\\ k\to 0}} f_y(a+h,b+\theta k)=f_y(a,b)$$

$\Rightarrow$ *if* $f_y(a+h,b+\theta k)-f_y(a,b)=\eta$, then $\eta\to 0$ as $h\to 0$, $k\to 0$.

So $f_y(a+h,b+\theta k)=f_y(a,b)+\eta$ & *So*

$$f(a+h,b+k)-f(a,b)=hf_x(a,b)+kf_y(a,b)+\varepsilon h+\eta k$$

where $\varepsilon\to 0, \eta\to 0$, as $(h,k)\to(0,0)$

$\Rightarrow f$ is differentiable at (a, b).

Remarks : (Alternative set of condition)

If (i) f_y exists at (a, b) & (ii) f_x is continuous at (a, b) then f is differentiable at (a, b).

Note : The condition of continuity of one of the partial derivatives at the point is sufficient only, but not necessary.

Example :

$$f(x,y)=\begin{cases} x^2\sin\frac{1}{x}+y^2\sin\frac{1}{y}, & xy\neq 0\\ x^2\sin\frac{1}{x}, & x\neq 0, y=0\\ y^2\sin\frac{1}{y}, & x=0, y\neq 0\\ 0 & x=0=y\end{cases}$$

As $\lim_{t\to 0} t\sin\frac{1}{t}=0$, *so* $f_x(0,0)=0, f_y(0,0)=0$

$$f_x(x,y)=\begin{cases}2x\sin\frac{1}{x}-\cos\frac{1}{x}, x\neq 0\\ 0,\ x=0\end{cases} \quad \& \quad f_y(x,y)=\begin{cases}2y\sin\frac{1}{y}-\cos\frac{1}{y}, y\neq 0\\ 0,\ y=0\end{cases}$$

As $\lim_{t\to 0}\cos\frac{1}{t}$ does not exist so neither f_x nor f_y is continuous at (0, 0).

For differentiability at (0, 0), we must have

$f(0+h, 0+k) - f(0,0) = hf_x(0,0) + kf_y(0,0) + h\varphi + k\psi$

where $\varphi \to 0, \psi \to 0, h \to 0, k \to 0.$

$\Rightarrow h^2 \sin\frac{1}{h} + k^2 \sin\frac{1}{k} - 0 = h.0 + k0 + h\varphi(h,k) + k\psi(h,k)\ etc.$

We take $\varphi(h,k) = \begin{cases} h\sin\frac{1}{h}, h \neq 0 \\ 0,\ h = 0 \end{cases}$

$\psi(h,k) = \begin{cases} k\sin\frac{1}{k}, k \neq 0 \\ 0,\ k = 0 \end{cases}$

Then $\varphi \to 0\ as\ h \to 0\ \&\ \psi \to 0\ as\ k \to 0$

So $f(x,y)$ is differentiable at (0,0) though neither f_x nor f_y is continuous at (0,0).

7.5 Total Differential

If $f(x, y)$ be a differentiable function, then $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ is called the total differential df of f.

$\Delta f = f(x+\Delta x, y+\Delta y) - f(x,y)$ is the total increment.

$\Delta x = dx, \Delta y = dy$ but $\Delta f \neq df$.

Example : If $f(x,y) = x^2 + xy + y^2 - 4\ \ln x - 10\ln y$, find $df(1,2)$

Here $f_x = 2x + y - \frac{4}{x}, f_y = x + 2y - \frac{10}{y}$ &

So at (1, 2), $f_x = 0, f_y = 0\ \&\ df(1,2) = 0.$

Examples :

(1) Let $f(x,y) = |xy|^p, (x,y) \in \mathbb{R}^2$

Show that f is differentiable at (0, 0) only if $p > \frac{1}{2}$

First we note that if $p < 0$, f cannot be defined at (0, 0) : So $p \geq 0$

For p > 0, $\lim_{h\to 0} \frac{f(0+h, 0) - f(0,0)}{h} = 0, \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k} = 0$

So $f_x(0,0) = 0 = f_y(0,0)$

In order to be differentiable at (0, 0), We must have

$f(0+h, 0+k) - f(0,0) = h.f_x(0,0) + kf_y(0,0) + \varepsilon_1 h + \varepsilon_2 k$

where $\varepsilon_1 \to 0, \varepsilon_2 \to 0$ as $h \to 0$, $k \to 0$.

$\Rightarrow |hk|^p = \varepsilon_1 h + \varepsilon_2 k$ where $\varepsilon_1 \to 0, \varepsilon_2 \to 0$ as $(h,k) \neq (0,0)$

In particular, if $k = h$ $(\neq 0)$

$|h|^{2p} = h\{\varepsilon_1(h,h) + \varepsilon_2(h,h)\}$

$\Rightarrow \pm(|h|)^{2p-1} = \varepsilon_1(h,h) + \varepsilon_2(h,h)$

Note that if $h \to 0$, $RHS \to 0$ but $LHS \to 0$ only when $2p - 1 > 0$ i.e. $p > \frac{1}{2}$

So when $p \leq \frac{1}{2}$ f cannot be differentiable at (0, 0).

Let us consider the case $p > \frac{1}{2}$

We take $\varepsilon_1(h,k) = \begin{cases} \frac{|hk|^p}{2h}, h \neq 0 \\ 0, h=0 \end{cases}$ & $\varepsilon_2(h,k) = \begin{cases} \frac{|hk|^p}{2k}, k \neq 0 \\ 0, k=0 \end{cases}$

$$\left|\varepsilon_1(h,k)-0\right| = \frac{|hk|^p}{2|h|} = \frac{1}{2}|hk|^{p-1}|k| \le \frac{1}{2}\left(\frac{h^2+k^2}{2}\right)^{p-1}(h^2+k^2)^{1/2}$$

$$\Rightarrow |\varepsilon_1(h,k)-0| \le \frac{1}{2^p}(h^2+k^2)^{p-\frac{1}{2}}$$

Let $\varepsilon > 0$ be any number

$|\varepsilon_1(h,k)-0| < \varepsilon$ whenever $0 < \sqrt{(h^2+k^2)} < \delta$

$$\delta = (2^p.\varepsilon)^{1/(2p-1)}, \ p > \frac{1}{2}$$

So $\varepsilon_1 \to 0$ as $(h,k) \to (0,0)$

Similarly $\varepsilon_2 \to 0 \; as \; (h,k) \to (0,0)$

Thus, $f(x,y)$ is differentiable at (0,0) when $p > \frac{1}{2}$.

(2) Let $f(x,y) = \begin{cases} \dfrac{x^6-2y^4}{x^2+y^2}, & x^2+y^2 \ne 0 \\ 0, & x^2+y^2 = 0 \end{cases}$

Show that f is differentiable at (0, 0).

Simple computation shows that $f_x(0,0) = 0 = f_y(0,0)$.

In order to be differentiable at (0,0), we must have

$$f(0+h,0+k) - f(0,0) = h.f_x(0,0) + kf_y(0,0) + h\varphi(h,k) + k\psi(h,k)$$

where $\varphi \to 0, \psi \to 0 \; as \; (h,k) \to (0,0)$.

$$\Rightarrow \frac{h^6-2k^4}{h^2+k^2} - 0 = h.0 + k.0 + h\varphi(h,k) + k\psi(h,k) etc.$$

We take $\varphi(h,k) = \begin{cases} \dfrac{h^5}{h^2+k^2}, & h^2+k^2 \ne 0 \\ 0, & h^2+k^2 = 0 \end{cases}$ & $\psi(h,k) = \begin{cases} \dfrac{-2k^3}{h^2+k^2}, & h^2+k^2 \ne 0 \\ 0, & h^2+k^2 = 0 \end{cases}$

Let $\varepsilon > 0$ be any number

Taking $h = r\cos\theta,\ k = r\sin\theta$

$$|\varphi(h,k) - 0| = |r^3\cos^5\theta| \le |r|^3 < \varepsilon$$

whenever $\sqrt{(x^2+y^2)} < \varepsilon^{\frac{1}{3}} \Rightarrow \lim_{(h,k)\to(0,0)} \varphi(h,k) = 0$

Exactly in a similar way, $\lim_{(h,k)\to(0,0)} \psi(h,k) = 0$

So f is differentiable at (0,0)

(3) Let $f(x,y) = \begin{cases} (x^2+y^2)\sin\dfrac{1}{\sqrt{(x^2+y^2)}}, & x^2+y^2 \ne 0 \\ 0, & x^2+y^2 = 0 \end{cases}$

Show that f is differentiable at (0,0) though f_x, f_y are not continuous at (0, 0).

$$\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} h\sin\frac{1}{|h|} = 0 \ \&$$

$$\lim_{k\to 0}\frac{f(0,0+k)-f(0,0)}{k} = \lim_{k\to 0} k\sin\frac{1}{|k|} = 0$$

In order to be differentiable at (0, 0), we must have

$$f(0+h,\ 0+k) - f(0,0) = hf_x(0,0) + kf_y(0,0) + h\varphi(h,k) + k\psi(h,k)$$

where $\varphi \to 0, \psi \to 0$ as $h \to 0, k \to 0$

$$(h^2+k^2)\sin\frac{1}{\sqrt{(h^2+k^2)}} - 0 = h\cdot 0 + k\ 0 + h\varphi(h,k) + k\psi(h,k) \text{ etc.}$$

We take $\varphi(h,k) = \begin{cases} h\sin\dfrac{1}{\sqrt{(h^2+k^2)}}, & h^2+k^2 \ne 0 \\ 0, & h^2+k^2 = 0 \end{cases}$ &

$$\psi(h,k)=\begin{cases} k\sin\dfrac{1}{\sqrt{(h^2+k^2)}}, h^2+k^2\neq 0 \\ 0,\ h^2+k^2=0 \end{cases}$$

So, $\varphi\to 0, \psi\to 0\ as\ h\to 0, k\to 0$.

Hence f is differentiable at (0, 0).

We know that

$$f_x(x,y)=\begin{cases} 2x\sin\dfrac{1}{\sqrt{(x^2+y^2)}}-\dfrac{x}{\sqrt{(x^2+y^2)}}\cos\dfrac{1}{\sqrt{x^2+y^2}}, x^2+y^2\neq 0 \\ 0, x^2+y^2=0 \end{cases}$$

when $x > 0$, $f_x(x,x)=2x\sin\dfrac{1}{x\sqrt{2}}-\dfrac{1}{\sqrt{2}}\cos\dfrac{1}{x\sqrt{2}}$

$\Rightarrow \lim\limits_{x\to 0} f_x(x,x)$ does not exist.

$\Rightarrow f_x$ is not continuous at (0,0). Similarly f_y is not continuous at (0,0).

(4) Let $f(x,y)=(|x+y|+x+y)^k, (x,y)\in\mathbb{R}^2$.

Show that f is differentiable for all $(x,y)\in\mathbb{R}^2$ if $k>1$.

We have deduced earlier that f_x, f_y exist at (0, 0) only if $k > 1$.

When $k>1, f_x(x,y)=f_y(x,y)=\begin{cases} k.2^k(x+y)^{k-1} \\ 0, if\ x+y\le 0 \end{cases}$, if $x+y>0$

Both f_x and f_y are continuous at each point $(x,y)\in\mathbb{R}^2$ as $(x+y)^{k-1}$ is also so.

So f is differentiable for all $(x,y)\in\mathbb{R}^2$ if $k > 1$.

(5) Let $f(x,y)=\begin{cases} x\sin(4\tan^{-1}\dfrac{y}{x}), \text{ for } x>0 \\ 0, \text{ for } x=0 \,\&\, \text{ for all } y \end{cases}$

Verify the following properties of f at the point (0, 0) :

(i) $\frac{\partial}{\partial x} f(0, y)$ is continuous w.r.t. y

(ii) $\frac{\partial}{\partial y} f(x, 0)$ is discontinuous w.r.t. x

(iii) f is not differentiable at (0,0)

Note that $f_x(0, y) = \lim_{h \to 0+} \frac{f(h, y) - f(0, y)}{h} = \lim_{h \to 0+} \sin\left(4\tan^{-1}\frac{y}{h}\right) = 0$

$$f_x(0,0) = \lim_{h \to 0+} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0+} \frac{0-0}{h} = 0.$$

So $\lim_{y \to 0} f_x(0, y) = f_x(0,0) \Rightarrow f_x(0, y)$ is continuous w.r.t. y at (0, 0)

$$f_y(x,0) = \lim_{k \to 0} \frac{f(x,k) - f(x,0)}{k} = \lim_{k \to 0} \frac{xSin(4\tan^{-1}\frac{k}{x})}{k} \left(\frac{0}{0}\right)$$

$$= \lim_{k \to 0} \frac{4x^2 \cos(4\tan^{-1}\frac{k}{x})}{1(x^2+k^2)} = 4 \quad \text{(by L. Hopital's rule)}$$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} \text{ (if it exists)} = \lim_{k \to 0} \frac{0-0}{k} = 0$$

So $\lim_{x \to 0+} f_y(x,0) \neq f_y(0,0) \Rightarrow f_y(x,0)$ is discontinuous w.r.t. x at (0, 0)

In order to be differentiable at (0,0), we must have

$$f(0+h, 0+k) - f(0,0) = hf_x(0,0) + kf_y(0,0) + \varepsilon_1 h + \varepsilon_2 k$$

where $\varepsilon_1 \to 0, \varepsilon_2 \to 0$ as $(h,k) \to (0,0)$.

$\Rightarrow h\sin(4\tan^{-1}\frac{k}{h}) - 0 = h.0 + k.0 + \varepsilon_1 h + \varepsilon_2 k$ where ......

For $k = 2h$, $h\sin(4\tan^{-1}2) = (\varepsilon_1 + 2\varepsilon_2)h$

$h \neq 0$ so $\sin(4\tan^{-1}2) = \varepsilon_1 + 2\varepsilon_2$

As $h \to 0$, $RHS \to 0$, $LHS \neq 0$.

So f is not differentiable at (0,0).

Differentiability of composite function & Chain Rule

Theorem (1) : Let

(i) $x = \varphi(u,v), y = \psi(u,v)$ be two functions of u, v defined in a domain $S \subset \mathbb{R}^2$ and differentiable at point (u, v) of S

(ii) $z = f(x,y)$ be defined on $S_1 \subset \mathbb{R}^2$ and differentiable at (x, y) of S_1.

(iii) S_1 be the image set of S. Then z, defined as a function of u, v, is differentiable at the corresponding point (u, v).

Proof. As $z = f(x,y)$ is differentiable, so

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y) = f_x.\Delta x + f_y.\Delta y + \varepsilon_1.\Delta x + \varepsilon_2.\Delta y \qquad (1)$$

where $\varepsilon_1 \to 0, \varepsilon_2 \to 0$ as $\Delta x \to 0, \Delta y \to 0$

As $x = \varphi(u,v), y = \psi(u,v)$ are differentiable functions, so

$$\Delta x = \varphi(u+\Delta u, v+\Delta v) - \varphi(u,v) = \varphi_u \Delta u + \Delta v.\varphi_v + \varepsilon_3 \Delta u + \varepsilon_4 \Delta v$$

where $\varepsilon_3 \to 0, \varepsilon_4 \to 0$ *as* $\Delta u \to 0, \Delta v \to 0$ (2)

Also $\Delta y = \psi(u+\Delta u, v+\Delta v) - \psi(u,v) = \psi_u \Delta u + \psi_v \Delta v + \varepsilon_5.\Delta u + \varepsilon_6 \Delta v$ (3)

where $\varepsilon_5 \to 0, \varepsilon_6 \to 0$ *as* $\Delta u \to 0, \Delta v \to 0$

Consequently, by (1), (2) & (3), we have

$$\Delta z = (f_x\varphi_u + f_y\varphi_u)\Delta u + (f_x\varphi_v + f_y\psi_v)\Delta v + (\varepsilon_3 f_x + \varepsilon_1\varphi_u + \varepsilon_1\varepsilon_3 + f_y\varepsilon_5 + \varepsilon_2\psi_u$$

$$+\varepsilon_2\varepsilon_5)\,\Delta u + (f_x\varepsilon_4 + \varepsilon_1\varphi_v + \varepsilon_1\varepsilon_4 + f_y.\varepsilon_6 + \varepsilon_2\psi_v + \varepsilon_2\varepsilon_6)\Delta v$$

$$= (f_x\varphi_u + f_y\psi_u)\Delta u + (f_x\varphi_v + f_y\psi_v)\Delta v + \eta_1\Delta_u + \eta_2.\Delta v$$

We know that differentiability $\Rightarrow$ continuity, so as $\Delta u \to 0, \Delta v \to 0$,

we have $\Delta x \to 0, \Delta y \to 0$. Therefore as $\Delta u \to 0, \Delta v \to 0$,

we get $\eta_1 \to 0, \eta_2 \to 0$

So $z = F(u,v)$ is differentiable function of u & v. Also

$$\left.\begin{aligned}\frac{\partial F}{\partial u} &= \frac{\partial f}{\partial x}.\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}.\frac{\partial y}{\partial u}\\ \frac{\partial F}{\partial v} &= \frac{\partial f}{\partial x}.\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}.\frac{\partial y}{\partial v}\end{aligned}\right\}$$

The last two results are known as 'Chain rule'.

Theorem 2 : Let in theorem 1, x and y be differentiable functions of single variable t so that the composite function $z = f(\varphi(t), \psi(t)) = F(t)$ can be defined, then $F(t)$ is differentiable function and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}.\frac{dx}{dt}+\frac{\partial f}{\partial y}.\frac{dy}{dt}$$

Here $dx = \varphi'(t)\Delta t,\ dy = \psi'(t)\Delta t$ putting in (1) of theorem 1, we get

$$\Delta z = f_x\varphi'(t)\Delta t + f_y\psi'(t)\Delta t + \varepsilon_1\varphi'(t)\Delta t + \varepsilon_2\psi'(t)\Delta t$$

Note that $\Delta x = \varphi(t+\Delta t)-\varphi(t),\ \Delta y = \psi(t+\Delta t)-\psi(t)$

Here φ, ψ are continuous functions of t and so as

$$\Delta t \to 0, \Delta x \to 0, \Delta y \to 0 \Rightarrow \varepsilon_1 \to 0, \varepsilon_2 \to 0$$

Therefore $z = f(t)$ is differentiable function of t &

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt}+\frac{\partial z}{\partial y}.\frac{dy}{dt} \quad \text{(Chain Rule).}$$

An important note

The differentiability of the concerned functions or the continuity of the first order partial derivatives can not be dropped.

Illustration Let $z = f(x,y) = \begin{cases}\dfrac{x^2y}{x^2+y^2}, & x^2+y^2 \neq 0\\ 0, & x^2+y^2=0\end{cases}$

Here $f_x = \begin{cases}\dfrac{2xy^3}{(x^2+y^2)^2}, & x^2+y^2 \neq 0\\ 0, & x^2+y^2=0\end{cases}$ & $f_y = \begin{cases}\dfrac{x^2(x^2-y^2)}{(x^2+y^2)^2}, & x^2+y^2 \neq 0\\ 0, & x^2+y^2=0\end{cases}$

In case of both f_x and f_y, if we take $(x,y) \to (0,0)$ along $y = mx$, we can infer that neither f_x nor f_y is continuous at (0, 0).

Also $f(0 + h, 0 + k) - f(0,0) = hf_x(0, 0) + kf_y(0,0) + \varepsilon_1 h + \varepsilon_2 k$

where $\varepsilon_1 \to 0, \varepsilon_2 \to 0$ as $(h, k) \to (0,0)$ must hold for differentiability at (0,0).

$\Rightarrow \dfrac{h^2k}{h^2+k^2} = h.0 + k0 + \varepsilon_1 h + \varepsilon_2 k$ etc. In particular, if $h = k$,

$\dfrac{h^3}{2h^2} = h(\varepsilon_1 + \varepsilon_2) \Rightarrow \dfrac{1}{2} = \varepsilon_1 + \varepsilon_2$. But as $h \to 0$, RHS $\to 0$, LHS $\neq 0$.

So f is not differentiable at (0,0).

We introduce a new variable 't' by setting $x = y = t$. We have a composite function of t.

By chain rule, $\dfrac{du}{dt} = u_x.x_t + u_y.y_t = 0$

But inserting $x = y = t$ in $f(x,y)$, we get $u = \dfrac{1}{2}t$ for all t. So

$\dfrac{du}{dt} = \dfrac{1}{2}$. Hence $\dfrac{du}{dt} = u_x.x_t + u_y.y_t$ does not hold.

Note : (Caution) We have seen in case of first order derivative for function of one variable, $\dfrac{dy}{dx} = 1\Big/\dfrac{dx}{dy}$ or $\dfrac{dx}{dy} = 1\Big/\dfrac{dy}{dx}$. Here $\dfrac{dy}{dx} \neq 0$, $\dfrac{dx}{dy} \neq 0$

This type of relation does not hold in case of function of two variables.

Let $x = r\cos\theta, y = r\sin\theta$ and so $r^2 = x^2 + y^2$

$$\Rightarrow 2r\frac{\partial r}{\partial x} = 2x \ \& \ 2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \frac{\partial r}{\partial y} = \sin\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta, \frac{\partial y}{\partial r} = \sin\theta \ \& \text{ So } \frac{\partial r}{\partial x} \neq 1\Big/\frac{\partial x}{\partial r}, \ \frac{\partial r}{\partial y} \neq 1\Big/\frac{\partial y}{\partial r}.$$

The relation between $\frac{\partial x}{\partial \theta} \& \frac{\partial \theta}{\partial x}, \frac{\partial y}{\partial \theta} \& \frac{\partial \theta}{\partial y}$ can also be examined.

Examples :

(1) Transform the following equation to new independent variables u, v.

$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$, if $u = x$, $v = x^2 + y^2$.

By chain rule $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial x} = z_u.1 + z_v.2x$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial y} = z_u.0 + z_v.2y$$

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = yz_x + 2xyz_v - 2xyz_v = yz_u$$

So $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$ is transformed into $\frac{\partial z}{\partial u} = 0$

(2) A function z of the arguments x and y is defined by the equation

$x = u + v, y = u^2 + v^2, z = u^3 + v^3 \quad (u \neq v)$

Show that $\frac{\partial z}{\partial x} = -3uv, \frac{\partial z}{\partial y} = \frac{3}{2}(u + v)$

Here taking differentials, $dx = du + dv$, $dy = 2udu + 2vdv$ & $dz = 3u^2du + 3v^2dv$.

From the first two relations

$$dy - 2vdx = (2udu + 2vdv) - (2vdu + 2vdv) = 2(u - v)du$$

$$\Rightarrow du = \frac{-2v}{2(u - v)}dx + \frac{1}{2(u - v)}dy.$$

Again $dy - 2udx = (2udu + 2vdv) - (2udu + 2udv) = 2(v - u)dv$.

$$\Rightarrow dv = \frac{2u}{2(u - v)}dx - \frac{1}{2(u - v)}dy.$$

So, $dz = 3u^2\left\{\frac{-v}{u-v}dx + \frac{1}{2(u-v)}dy\right\} + 3v^2\left\{\frac{u}{u-v}dx - \frac{1}{2(u-v)}dy\right\}$

$\equiv \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$

As x and y are independent variables, we get

$$\frac{\partial z}{\partial x} = \frac{-3u^2v + 3v^2u}{u-v} = -3uv \;\&\; \frac{\partial z}{\partial y} = \frac{3u^2 - 3v^2}{2(u-v)} = \frac{3}{2}(u+v)$$

(3) If the relationship $u = f(x, y)$, $v = g(x, y)$ where f & g are differentiable functions of x and y, specify x and y as differentiable functions of u & v, prove that

$$(u_x v_y - u_y v_x)(x_u y_v - x_v y_u) = 1$$

$$u = f(x, y) \Rightarrow du = f_x dx + f_y dy = u_x dx + u_y dy$$

$$\& \;\; v = g(x, y) \Rightarrow dv = g_x dx + g_y dy = v_x dx + v_y dy$$

From these two, $dx = \frac{1}{J}(v_y du - u_y dv) \;\&\; dy = \frac{1}{J}(-v_x du + u_x dv)$ where

$J = u_x v_y - u_y v_x \neq 0$

By hypothesis, x & y are differentiable functions of u & v, so

$dx = x_u du + x_v dv, dy = y_u du + y_v dv$ & so

$$x_u = \frac{v_y}{J}, x_v = \frac{-u_y}{J}, \; y_u = \frac{-v_x}{J}, \; y_v = \frac{u_x}{J}$$

So $\left(u_x v_y - u_y v_x\right)\left(x_u y_v - x_v y_u\right) = \left(u_x v_y - u_y v_x\right)\left\{\frac{(v_y u_x - u_y v_x)}{J^2} = \frac{J^2}{J^2} = 1.\right.$

(4) If $x = cuv, y = c\left\{(1+u^2)(1-v^2)\right\}^{1/2}$ where c is a non-zero constant. show that $\frac{1}{y}\left\{y\frac{\partial V}{\partial x} - x\frac{\partial V}{\partial y}\right\} = \frac{\left(v\frac{\partial V}{\partial u} + u\frac{\partial V}{\partial v}\right)}{c(u^2+v^2)}$ given that V is any differentiable function of x and y.

By chain rule, $\frac{\partial V}{\partial u}=\frac{\partial V}{\partial x}.\frac{\partial x}{\partial u}+\frac{\partial V}{\partial y}.\frac{\partial y}{\partial u}=V_x.cv+V_y.\frac{c(1-v^2)^{1/2}.2u}{2(1+u^2)^{1/2}}$

$$\frac{\partial V}{\partial v}=\frac{\partial V}{\partial x}.\frac{\partial x}{\partial v}+\frac{\partial V}{\partial y}.\frac{\partial y}{\partial v}=V_x.c_u+V_y.\frac{c\left(1+u^2\right)^{1/2}(-2v)}{2(1-v^2)^{1/2}}$$

So $vV_u+uV_v=c(u^2+v^2)[V_x-\frac{x}{y}V_y]\Rightarrow\frac{vV_u+uV_v}{c(u^2+v^2)}=\frac{1}{y}\left\{yV_x-xV_y\right\}$.

(5) Show that whatever the differentiable function φ is, the relationship

$$\varphi(cx-az,cy-bz)=0 \text{ implies that } a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}=c.$$

By hypothesis, the given relation defines z as a differentiable function of x & y. We put $cx-az=p, cy-bz=q$

So $\varphi(p,q)=0\Rightarrow d\varphi=\varphi_p dp+\varphi_q dq=0$

$$\Rightarrow\varphi_p(cdx-adz)+\varphi_q(cdy-bdz)=0\Rightarrow dz=\frac{c\varphi_p dx+c\varphi_q dy}{a\varphi_p+b\varphi_q}=z_x dx+z_y dy$$

So, $z_x=\frac{c\varphi_p}{a\varphi_p+b\varphi_q}$, $z_y=\frac{c\varphi_q}{a\varphi_p+b\varphi_q}$. Hence $az_x+bz_y=c$.

(6) By putting $G = x^n H$ & changing the independent variables x, y to u, v where $u=\frac{y}{x}$, $v=xy$; transform the equation $x\frac{\partial G}{\partial x}+y\frac{\partial G}{\partial y}=nG$.

Hence show that $G=x^n\varphi\left(\frac{y}{x}\right)$ where φ denotes arbitrary function

By chain rule, $\frac{\partial H}{\partial x}=\frac{\partial H}{\partial u}.\frac{\partial u}{\partial x}+\frac{\partial H}{\partial v}.\frac{\partial v}{\partial x}=\frac{-y}{x^2}.H_u+yH_v$

& $\frac{\partial H}{\partial y}=\frac{\partial H}{\partial u}.\frac{\partial u}{\partial y}+\frac{\partial H}{\partial v}.\frac{\partial v}{\partial y}=\frac{1}{x}.H_u+xH_v$

Given $G=x^nH$, So $G_x=nx^{n-1}H+x^n.H_x$ & $G_y=x^nH_y$

So, $xG_x + yG_y = nG$ is transformed into

$$nx^nH + x^{n+1}H_x + x^n yH_y = nx^nH \Rightarrow xH_x + yH_y = 0$$

Consequently, $-\frac{y}{x}H_u + xyH_v + \frac{y}{x}H_u + xyH_v = 0 \Rightarrow H_v = 0$

So H is independent of v & we get $H = \varphi(u)$ where φ is arbitrary function

Therefore $G = x^nH\left(\frac{y}{x}\right)$

(7) Transform the equation $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = (y-x)z$, by introducing new independent variables $u = x^2 + y^2, v = \frac{1}{x} + \frac{1}{y}$ and the new function

$$w = \ln z - (x+y).$$

Taking differentials, $du = 2xdx + 2ydy, dv = -\frac{1}{x^2}dx - \frac{1}{y^2}dy$ &

$$dw = \frac{dz}{z} - (dx + dy).$$

Note that $dw = \frac{\partial w}{\partial u}du + \frac{\partial w}{\partial v}dv = w_u.(2xdx + 2ydy) + w_v\left(-\frac{dx}{x^2} - \frac{dy}{y^2}\right)$

$$= \left(2xw_u - \frac{w_v}{x^2}\right)dx + \left(2yw_u - \frac{w_v}{y^2}\right)dy$$

$$= \frac{dz}{z} - (dx + dy).$$

So, $dz = (2xzw_u - \frac{z}{x^2}w_v + z)dx + (2yzw_u - \frac{z}{y^2}w_v + z)dy$

$$\equiv \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

So $\frac{\partial z}{\partial x} = 2xzw_u - \frac{z}{x^2}w_v + z, \frac{\partial z}{\partial y} = 2yzw_u - \frac{z}{y^2}w_v + z$

Putting in given equation $yz_x - xz_y = (y-x)z$, we get

$$(2xyzw_u - \frac{yz}{x^2}w_v + yz) - (2xyz\, w_u - \frac{xz}{y^2}w_v + xz) = (y-x)z$$

$$\Rightarrow w_v = 0$$

So the transformed equation is $\frac{\partial w}{\partial v} = 0$.

(8) Transform the equation $x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = z^2$, taking $u = x+y, v = \frac{1}{y} - \frac{1}{x}$

for new independent variables and $w = \frac{1}{z} - \frac{1}{x}$ for the new function

Taking differentials, $du = dx + dy, dv = \frac{-dy}{y^2} + \frac{dx}{x^2}, dw = \frac{-dz}{z^2} + \frac{dx}{x^2}$

$$dw = \frac{\partial w}{\partial u}du + \frac{\partial w}{\partial v}dv = w_u(dx+dy) + w_v\left(\frac{dx}{x^2} - \frac{dy}{y^2}\right) = \frac{-dz}{z^2} + \frac{dx}{x^2}$$

$$\Rightarrow dz = z^2\left(\frac{1}{x^2} - w_u - \frac{1}{x^2}w_v\right)dx + \frac{z^2}{y^2}w_v \cdot dy \equiv z_x dx + z_y dy$$

$$\Rightarrow z_x = \frac{z^2}{x^2} - z^2 w_u - \frac{z^2}{x^2}w_v \;\;\&\;\; z_y = \frac{z^2}{y^2}w_v$$

$x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = z^2$ is transformed into

$$z^2 - x^2z^2w_u - z^2w_v + z^2w_v = z^2 \Rightarrow w_u = 0,$$

the transformed equation is $\frac{\partial w}{\partial u} = 0$.

(9) Let $f(x,y) = \begin{cases} x^{4/3}\sin\frac{y}{x} & x \neq 0 \\ 0, x = 0 \end{cases}$

Find the points at which f is differentiable.

Hence $f_x(x,y) = \begin{cases} \frac{4}{3}x^{1/3}\sin\frac{y}{x} - yx^{-\frac{2}{3}}\cos\frac{y}{x}, \ x \neq 0 \\ 0, x = 0 \end{cases}$

$f_y(x,y) = \{x^{1/3}\cos\frac{y}{x}, x \neq 0$

$0, x = 0$

Due to the continuity of f_x, f_y, at $\mathbb{R}^2 \setminus \{(0,y) : y \in \mathbb{R}\}$,
using sufficient condition of differentiability, f is differentiable in $\mathbb{R}^2 \setminus \{(0,y) : y \in \mathbb{R}\}$.

(10) If $z = \sqrt{|xy|}$, $x = t$, $y = t + t^3$ where t > 0, verify that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt}$$

does not hold at t = 0, Explain why.

Here $\frac{dx}{dt} = 1, \frac{dy}{dt} = 1 + 3t^2$

$\lim_{h \to 0}\frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0}\frac{0-0}{h} = 0 \Rightarrow f_x(0,0) = 0$, Similarly $f_y(0,0) = 0$

Therefore, at $t = 0$, $\frac{dx}{dt}.\frac{\partial z}{\partial x} + \frac{dy}{dt}.\frac{\partial z}{\partial y} = 0.$

Putting x & y in z, $z = \sqrt{|t^2 + t^4|} = t\sqrt{(1+t^2)}$

$\frac{dz}{dt} = \sqrt{(1+t^2)} + \frac{t^2}{\sqrt{(1+t^2)}}$. At t = 0, $\frac{dz}{dt} = 1.$

So $\frac{dz}{dt} \neq \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt}$

We have discussed earlier that $|xy|^p$ is differentiable at (0,0) only when $p > \frac{1}{2}$.

So $\sqrt{|xy|}$ is not differentiable at (0,0) here & this is the reason for failure of chain rule here.

(11) Let $\varphi(x,y) = \begin{cases} x.\dfrac{x^4 - 4x^2y^2 - y^4}{(x^2+y^2)^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$

Show that φ is continuous at (0, 0), has a directional derivative in every direction at (0,0) but is not differentiable there.

$$\varphi(x,y) = \begin{cases} x\dfrac{(x^2-y^2)^2 - 2y^2(x^2+y^2)}{(x^2+y^2)^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$$

So when $x^2+y^2 \neq 0, \varphi(x,y) = x\left\{\left(\dfrac{x^2-y^2}{x^2+y^2}\right)^2 - \dfrac{2y^2}{x^2+y^2}\right\}$

So $|\varphi(x,y) - \varphi(0,0)| \leq 3|x| \leq 3(|x|+|y|)$ (1)

Let $\varepsilon > 0$ be any number. Then

$|\varphi(x,y) - \varphi(0,0)| < \varepsilon$ whenever $|x| < \delta, |y| < \delta$ & $\delta = \frac{\varepsilon}{6} \Rightarrow \varphi$ is continuous at (0,0).

(Also when $(x,y) \neq (0,0), \varphi(x,y)$ is the ratio of two continuous functions & so it is continuous at all such points).

For any $< a, b > \neq < 0, 0 >$ & for any $t \neq 0$,

$$\lim_{t\to 0}\frac{\varphi(ta,tb) - \varphi(0,0)}{t} = \frac{a(a^4 - 4a^2b^2 - b^4)}{a^2+b^2}$$

$\Rightarrow$ directional derivative exists in every direction at (0,0).

Putting a = 1, b = 0, we get $\varphi_x(0,0) = 1$ & putting a = 0, b = 1, we get

$\varphi_y(0,0) = 0$

For differentiability at (0, 0), we must have

$$\varphi(0+h, 0+k) - \varphi(0,0) = h\varphi_x(0,0) + k\varphi_y(0,0) + \varepsilon_1 h + \varepsilon_2 k$$

where $\varepsilon_1 \to 0, \varepsilon_2 \to 0$ as $h \to 0$, $k \to 0$.

$$\Rightarrow \frac{h(h^4 - 4h^2k^2 - k^4)}{(h^2+k^2)^2} - 0 = h.1 + k.0 + \varepsilon_1 h + \varepsilon_2 k \text{ etc.}$$

In particular, for $h = k$ ($\neq 0$), $RHS = h(1+ \varepsilon_1 + \varepsilon_2)$ &

$$\text{LHS} \Rightarrow \frac{h(h^4 - 4h^4 - h^4)}{(2h^2)^2} = \frac{-4h^5}{4h^4} = -h$$

So $-h = h(1+\varepsilon_1+\varepsilon_2) \Rightarrow -1 = 1+\varepsilon_1+\varepsilon_2$

As $h \to 0$, $RHS \to 1$, $LHS = -1$. So φ is not differentiable at (0,0)

Therefore, existence of directional derivative $\not\Rightarrow$ differentiability at that point.

(12) If $f(x,y)$ be differentiable, then f has a directional derivative in the direction of any unit vector $< a, b >$.

By hypothesis, if $g(h) = f(x_0 + ha, y_0 + hb)$, then

$$\lim_{h\to 0}\frac{g(h)-g(0)}{h} = \lim_{h\to 0}\frac{f(x_0+ha, y_0+hb) - f(x_0, y_0)}{h} = D_u f(x_0, y_0)$$

the directional derivative at (x_0, y_0) along unit vector $< a, b>$

Using differentiability of f ;

$$g'(h) = \frac{\partial f}{\partial x}.\frac{dx}{dh} + \frac{\partial f}{\partial y}.\frac{dy}{dh} = f_x(x_0, y_o).a + f_y(x_0, y_0).b$$

Hence the result follows.

7.6 Summary

In this unit, we have defined partial differentiation with some examples. We also studied directional derivative and total derivative. We have explained differentiability of composite function and studied the chain rule. The differentiability of a function of the variables at a point of its domain imply the existence of first order partial derivatives at that point but the converse is not true. This is a major departure from

the relevant result of function of one variable. Examples & results regarding this have been explained here.

7.7 Exercises

(1) Examine for the existence of double limit at (0, 0) :

(a) $f(x,y)=\dfrac{\sqrt{(x^2y^2+1)}-1}{x^2+y^2},\ x^2+y^2\neq 0$

(b) $f(x,y)=\dfrac{\sin(x^3+y^3)}{x^2+y^2},\ x^2+y^2\neq 0$

(c) $f(x,y)=\dfrac{x^2y^2}{x^4+y^4-x^2y^2},\ x^2+y^2\neq 0$

(2) Let $f(x,y)=\begin{cases}\exp\left\{\dfrac{-|x-y|}{x^2-2xy+y^2}\right\}, & x\neq y\\ 0 & x=y\end{cases}$

Show that f is continuous at (0, 0).

(3) Let $f(x,y)=\begin{cases}\dfrac{xy}{(x^2+y^2)^{5/2}}\left[1-\cos(x^2+y^2)\right], & (x,y)\neq(0,0)\\ K, & (x,y)=(0,0)\end{cases}$

Find K, if any, for which $f(x, y)$ is continuous at (0, 0).

(4) Show that the limit

$$\lim_{(x,y)\to(0,0)}\frac{1-\cos(x^2+y^2)}{(x^2+y^2)x^2y^2}$$

does not exist.

(5) Let $f(x,y)=\begin{cases}x^2y^2\log(x^2+y^2), & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

Show that f is differentiable at (0,0).

(6) Show that the expression

$(3x+y)dx+(x+3y)dy$ is a total differential of some function $u(x,y)$

& find $u(x,y)$.

(7) Let $f(x,y)=\begin{cases}\dfrac{x}{|y|}\cdot\sqrt{(x^2+y^2)}\,; y\neq 0\\ 0, y=0\end{cases}$

Examine for the existence of directional derivative at (0,0) & differentiability of f at (0,0).

(8) If $u=f(ax^2+2hxy+by^2)$ and $v=\varphi(ax^2+2hxy+by^2)$, show that

$$\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right)=\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right).$$

(9) If $u=f(x, y)$, $x=r\cos\theta, y=r\sin\theta$; then if the variables x, y are changed to r, θ, prove that

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

(10) Let $f(x,y)=\begin{cases}x, when\,|y|<|x|\\ -x, when\,|y|\geq|x|\end{cases}$

Examine whether f is differentiable at (0, 0).

(11) Let f be differentiable function where

$f(xy, z-2x)=0$

defines z as a differentiable function of x and y.

Show that under suitable condition (s) to be stated by you,

$$x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=2x$$

(12) Let z be a differentiable function of u and v where $u = x^2 - y^2 - 2xy$, $v = y$;

prove that the equation $(x+y)\dfrac{\partial z}{\partial x}+(x-y)\dfrac{\partial z}{\partial y}=0$ is equivalent to $\dfrac{\partial z}{\partial v}=0$

(13) Is f differentiable at the origin

$$f(x,y)=\begin{cases} xy.\dfrac{x^2-y^2}{\sqrt{(x^2+y^2)}}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

(14) Let $f(x,y)=\begin{cases} \dfrac{x^2y}{3x^2+y^3}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$

If $x = t = y$ for all $t \geq 0$, examine whether

$\dfrac{df}{dt}=\dfrac{\partial f}{\partial x}.\dfrac{dx}{dt}+\dfrac{\partial f}{\partial y}.\dfrac{dy}{dt}$ holds at $t = 0$

If not, explain why.

(15) Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by

$$f(x,y)=x^4-2x^3y-xy.$$

Let $L_1\equiv 2\dfrac{\partial}{\partial x}-3\dfrac{\partial}{\partial y}$ & $L_2\equiv\dfrac{\partial}{\partial x}\left(\dfrac{\partial}{\partial y}\right)$

Examine whether $L_1(L_2f)=L_2(L_1f)$ holds.

(16) Let $f(x,y)=(x^3+y^3)^{1/3}$

(i) find f_x and f_y at (0, 0) (ii) test for the continuity and differentiability of f at (0, 0).

(17) Let $f(x,y)=\begin{cases} y.\dfrac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$

Show that f is continuous at the origin, has a directional derivative in every direction at (0, 0) but is not differentiable at that point.

(18) Let $f(x,y)=\begin{cases} \dfrac{x}{|y|}.\sqrt{(x^2+y^2)}, & x\neq 0 \\ 0, & y=0 \end{cases}$

Examine for the existence of directional derivative at (0, 0) and differentiability of f at (0, 0).

(19) If z and u be functions of x and y defined by

$$\{z-\varphi(u)\}^2 = x^2(y^2-u^2), \{z-\varphi(u)\}.\varphi'(u) = ux^2$$

show that $\frac{\partial z}{\partial x}.\frac{\partial z}{\partial y} = xy$

(Compute differential dz in terms of dx & dy)

(20) If for all values of the parameter λ and for some constant n,

$F(\lambda x, \lambda y) = \lambda^n F(x,y)$ identically, where F is differentiable function of x & y, show that

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = nF(x,y).$$

(21) Find the boundary and interior, exterior for each of the following subsets of $\mathbb{R}^2$

(a) $A = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$

(b) $B = \{(x, y) \in \mathbb{R}^2 \mid x > 0 \text{ and } y \neq 0\}$

(c) $C = A \cup B$

(d) $D = \{(x, y) \in \mathbb{R}^2 \mid x \notin Q\}$

7.8 Further Readings

1. Demidovich, B : *Problems in Mathematical Analysis* (Mir Publishers).
2. Malik & Arora : *Mathematical Analysis* (Wiley Eastern).
3. Mukherjee S.K : *Advanced Differential Calculus of several variables* — (Fifth Edition) (Academic Publishers)
4. Wider David : *Advanced Calculus* (Prentice Hall).